DISTINGUISHED RINGS OF LINEAR TRANSFORMATIONS

By

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To my wife

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TABLE OF CONTENTS

Page

ACKNOWLEDGEMENTS iii

INTRODUCTION 1

Chapter

I. PRELIMINARY RESULTS 5

II. DISTINGUISHED RINGS AND THEIR ASSOCIATED GRAPHS AND LATTICES 8

III. A NORMAL FORM FOR DISTINGUISHED RINGS . . 21

IV. RICKART AND BAER RINGS 29

BIBLIOGRAPHY 53

BIOGRAPHICAL SKETCH 55

INTRODUCTION

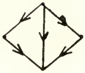
Rings in which the left (right) annihilator of any subset is generated as a left (right) ideal by an idempotent were first studied by Baer [13]. He noted that the ring of all linear transformations on an arbitrary vector space had this property. Such rings were called Baer rings by Kaplansky [3], wherein he remarked that the ring of all bounded operators in a Hilbert space is Baer. In his notes, "Rings of Operators" [3], Kaplansky gave an example of a non-semi-simple Baer ring: The ring of all 2 x 2 triangular matrices

$$\begin{Vmatrix} x & 0 \\ y & z \end{Vmatrix}$$

over a field. Wolfson [5], motivated by Kaplansky's example, showed that this is a particular case of a class of Baer subrings of the ring of all linear transformations on a vector space. In [5], he proved that if V is a vector space of arbitrary dimension over a division ring and $0 = V_o \subset V_1 \subset \ldots \subset V_n = V$ is a fixed chain of subspaces of V, then the ring of all linear transformations on V leaving each V_i in variant is Baer. The matrix representation of such a ring (with respect to a basis adapted to the V_i) is a complete, blocked triangular, matrix ring over a division ring.

Motivated, in part, by Wolfson's result, Johnson [2] introduced the notion of a distinguished ring of linear transformations: A ring $\mathfrak{A}$ of linear transformations of a vector space V over a division ring which contains every linear transformation on V which leaves invariant all the elements of the lattice J of invariant subspaces of $\mathfrak{A}$. Johnson confined his attention to distinguished rings for which the lattice J is finite as well as distributive. He established a rather complicated, necessary and sufficient condition for a left (right) ideal of a distinguished ring of linear transformations to be generated by an idempotent. He then showed that this condition is verified for all ideals which are left (right) annihilators in the special case where J is a chain. This gives him Wolfson's result as a corollary. Johnson also gave a necessary condition (on the lattice J) for a distinguished ring to be Baer. Clark's results (see below) translated into graph theory show that this condition is also sufficient when V is finite-dimensional.

Independently, Clark [1] generalized Wolfson's result for vector spaces of finite dimension taking another approach. He noted that each semigroup S of matrix units is uniquely determined by a transitive directed graph G(S) and proved that rings obtained by taking linear combinations of elements of S over a division ring are Baer if and only if (i) the graph G(S) has a loop at every non-isolated vertex and (ii) G(S) has no section graph isomorphic to the following graph:



In this manuscript, we merge the approaches taken by Clark and Johnson to the problem of characterizing Baer rings of linear transformations. A definition of a distinguished ring of linear transformations on a vector space V over a division ring D (see Definition 2.1), distinct from that of Johnson, is given such that the lattice J of invariant subspaces is always finite and distributive. It is shown that the new definition is equivalent to that of Johnson if the lattice J is finite and D has characteristic zero, and to that of Clark if the dimension of V is finite. It is also shown that such rings are determined by a decomposition of V into subspaces together with a reflexive, transitive, directed graph on a vertex set with cardinality equal to the number of subspaces in the decomposition of V. A partial generalization of Clark's result is made. We show that any distinguished ring $\mathfrak{A}$ (in the sense of Definition 2.1) is a Rickart ring (a ring in which the left (right) annihilator of a single element is generated as a left ideal by an idempotent) if and only if the graph of $\mathfrak{A}$ has no subgraph isomorphic to the displayed graph above.

In Chapter I, we prove some intuitively obvious facts concerning block multiplication of infinite dimensional row finite matrices.

In Chapter II, we introduce the alternate definition of distinguished ring referred to above. We show that this definition is equivalent to that of Johnson in case the lattice J of invariant subspaces is finite and distributive. We associate a finite, reflexive, transitive, directed graph with each distinguished ring such that J is finite and distributive and show that such a graph together with a decomposition of the underlying vector space determine a distinguished ring.

Chapter III deals with the triangularization of the graph associated with a distinguished ring $\mathfrak{A}$ and the corresponding normal form this implies for a matrix representation of $\mathfrak{A}$.

The main results as described above are contained in Chapter IV.

CHAPTER I

PRELIMINARY RESULTS

In this chapter we establish some elementary facts concerning the block multiplication of infinite dimensional row finite matrices.

Let V be a vector space of arbitrary dimension over a division ring D, and let $V = V_1 \oplus V_2 \oplus \ldots \oplus V_n$ where V_i is a subspace of V having a basis $\chi_i (i = 1, 2, \ldots, n)$. Let $\chi = \cup_{i=1}^{n} \chi_i$. By $\mathfrak{L}(V)$ we will mean the ring of all linear transformations on V. $\mathrm{Hom}_D(V_i, V_j)$ will, as usual, denote the set of all linear transformations from V_i into V_j. If S is a set, then #S will denote the cardinality of S, and $\#S < \infty$ will mean that S has finite cardinality.

Let M_D denote the set of all $\chi \times \chi$ row finite matrices over D. For

$(\alpha_{\lambda\nu})_{\lambda,\nu\epsilon\chi}$ and $(\beta_{\lambda\nu})_{\lambda,\nu\epsilon\chi}$ in M_D define

$$(\alpha_{\lambda\nu})_{\lambda,\nu\epsilon\chi} + (\beta_{\lambda\nu})_{\lambda,\nu\epsilon\chi} = (\alpha_{\lambda\nu} + \beta_{\lambda\nu})_{\lambda,\nu\epsilon\chi} \text{ and}$$

$$(\alpha_{\lambda\nu})_{\lambda,\nu\epsilon\chi} (\beta_{\lambda\nu})_{\lambda,\nu\epsilon\chi} = \Big(\sum_{\mu\epsilon\chi} a_{\lambda\mu}\beta_{\mu\nu}\Big)_{\lambda,\nu\epsilon\chi}$$

Under these operations M_D is a ring and is isomorphic to $\mathfrak{L}(V)$, see [7].

Each $(\alpha_{\lambda\nu})_{\lambda,\nu\epsilon\chi} \in M_D$ can be written in the form

$$A = \begin{Vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & & A_{nn} \end{Vmatrix}$$

where $A_{ij} = (\alpha_{\lambda\nu})_{\lambda\epsilon\chi_i, \nu\epsilon\chi_j}$ (i, j=1, 2, ..., n). Let $\bar{M}_D$ denote the collection of all such matrices. Define $A_{ik}\ B_{kj}$ $= (\alpha_{\lambda\nu})_{\lambda\epsilon\chi_i,\nu\epsilon\chi_k} (\beta_{\lambda\nu})_{\lambda\epsilon\chi_k,\nu\epsilon\chi_j} = \left(\sum_{\mu\epsilon\chi_k} \alpha_{\lambda\mu}\beta_{\mu\nu}\right)_{\lambda\epsilon\chi_i,\nu\epsilon\chi_j}$ and for $A = (A_{ij})$, $B = (B_{ij}) \epsilon \bar{M}_D$, where A_{ij} and B_{ij} are $\chi_i \times \chi_j$ row finite matrices over D, define $AB = (A_{ij})(B_{ij})$ $= (C_{ij}) = C$ where $C_{ij} = \sum_{k=1}^{M} A_{ik}\ B_{kj}$. C is in $\bar{M}_D$, for C is obtained from $(\alpha_{\lambda\nu})_{\lambda,\nu\epsilon\chi} (\beta_{\lambda\nu})_{\lambda,\nu\epsilon\chi} = \left(\sum_{\mu\epsilon\chi} \alpha_{\lambda\mu}\beta_{\mu\nu}\right)_{\lambda,\nu\epsilon\chi}$, since $C_{ij} = \left(\sum_{\mu\epsilon\chi} \alpha_{\lambda\mu}\beta_{\mu\nu}\right)_{\lambda\epsilon\chi_i,\nu\epsilon\chi_j}$. Defining addition in $\bar{M}_D$ in the obvious way, it can be shown that $\bar{M}_D$ is a ring and is isomorphic to M_D under the correspondence $(\alpha_{\lambda\nu})_{\lambda,\nu\epsilon\chi}$ $\rightarrow A = (A_{ij})$ if and only if $(\alpha_{\lambda\nu})_{\lambda\epsilon\chi_i,\nu\epsilon\chi_j}$ $= A_{ij}$ (i, j=1, 2, ..., n).

We need to show that if a matrix in $\bar{M}_D$ is partitioned into k x k blocked matrices and multiplication is defined in the obvious way, then the new blocked multiplication corresponds to the multiplication in $\bar{M}_D$. To accomplish this, let $\bar{M}_D$ be partitioned such that each element has the form

$$\begin{Vmatrix} A'_{11} & A'_{12} & \cdots & A'_{1k} \\ A'_{21} & A'_{22} & \cdots & A'_{2k} \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ A'_{k1} & A'_{k2} & \cdots & A'_{kk} \end{Vmatrix}$$

where A'_{ij} is a $p_i \times p_j$ blocked matrix $(i, j=1, 2, \ldots, k)$ and $n = p_1 + p_2 + \ldots + p_k$. Let A' and B' be two elements of $\overline{M}_D$ partitioned as in (1) and define

$$A'B' = (A'_{ij})(B'_{ij}) = \left(\sum_{t=1}^{k} A'_{it}B'_{tj}\right) \; (i, j=1, 2, \ldots, k)$$

where
$$A'_{it}B'_{tj} = \left(A_{ij}\right)_{j=1,2,\ldots,p_t}^{i=1,2,\ldots,p_i} \left(B_{ij}\right)_{j=1,2,\ldots,p_j}^{i=1,2,\ldots,p_i}$$
$$= \left(\sum_{s=1}^{p_t} A_{is}B_{sj}\right)_{j=1,2,\ldots,p_j}^{i=1,2,\ldots,p_i} .$$

Now if (A'_{ij}) and (B'_{ij}) $(i, j=1, 2, \ldots, k)$ are the blocked forms of (A_{ij}) and (B_{ij}) $(i, j=1, 2, \ldots, n)$, respectively, then

$$(C'_{ij}) = \left(\sum_{t=1}^{k} A'_{it}B'_{tj}\right) \; (i, j=1, 2, \ldots, k) \text{ is the}$$

blocked form of

$$(C_{ij}) = \left(\sum_{t=1}^{n} A_{it}B_{tj}\right) \; (i, j=1, 2, \ldots, n), \text{ since}$$

$n = p_1 + p_2 + \ldots + p_k$ and

$$\sum_{t=1}^{k} A'_{it}B'_{tj} = \sum_{t=1}^{k} \left(\sum_{s=1}^{p_t} A_{is}B_{sj}\right)_{j=1,2,\ldots,p_j}^{i=1,2,\ldots,p_i}$$
$$= \left(\sum_{s=1}^{n} A_{is}B_{sj}\right)_{j=1,2,\ldots,p_j}^{i=1,2,\ldots,p_i} .$$

CHAPTER II

DISTINGUISHED RINGS AND THEIR ASSOCIATED GRAPHS AND LATTICES

In this chapter we introduce the notion of a distinguished ring of linear transformations on a vector space V. We show that each such ring $\mathfrak{A}$ is essentially determined by a finite directed graph which in turn is determined by the lattice of $\mathfrak{A}$-submodules of V. In [2], Johnson introduced a similar but different class of rings which he called distinguished (see definition below). We show that Johnson's distinguished rings coincide with ours when the lattice of $\mathfrak{A}$-submodules of V is finite and distributive. In particular, this is the case when the underlying division ring has characteristic zero.

By a finite directed graph G on I_n, we simply mean a relation on I_n. The elements (i,j) of G will be called edges and those of I_n, vertices. We say that the edge (i,j) joins vertex i to vertex j. An edge (i,i) is called a loop. A finite directed graph G on I_n is said to be reflexive if $(i,i) \in \mathfrak{A}$ for each $i \in I_n$, i.e., there is a loop at each vertex, and transitive if $(i,j),(j,k) \in \mathfrak{A}$ implies $(i,k) \in \mathfrak{A}$.

Throughout the remainder of this paper V will denote a vector space of arbitrary dimension over a division ring D.

2.1 Definition. Let V be a vector space. Let $V = V_1 \oplus V_2 \oplus \ldots V_n$ be a direct decomposition of V and $\{J_i \mid i \in I_n\}$ be a family of subsets of I_n. Then a ring $\mathfrak{A} = \mathfrak{A}(V_i, J_i, n)$ of linear transformations on V is said to be a distinguished ring (henceforth a d-ring) if and only if for each $i \in I_n$ there is a $J_i \subset I_n$ with $i \in J_i$ such that

$$\text{(i)} \quad V_i\sigma \subset \sum_{j \in J_i} \oplus V_j \text{ for each } \sigma \in \mathfrak{A}, \text{ for each}$$

$i \in I_n$, and

$$\text{(ii)} \quad \text{if } \sigma \in \mathcal{L}(V) \text{ such that } V_i\sigma \subset \sum_{j \in J_i} \oplus V_j \text{ for}$$

each $i \in I_n$, then $\sigma \in \mathfrak{A}$.

Let $S = \{J_1, J_2, \ldots, J_n\} \subset \mathcal{P}(I_n)$. In order for $\mathfrak{A}(V_i, S, J_i)$ to be a d-ring, it is necessary and sufficient that $j \in J_i$ if and only if $J_j \subset J_i$, i.e., if and only if the relation $\sim$ on I_n defined by $i \sim j$ if and only if $j \in J_i$ is reflexive and transitive. These are easily verifiable modulo Lemmas 2.2 and 2.5.

2.2 Lemma. A decomposition $V = V_1 \oplus V_2 \oplus \ldots \oplus V_n$ of a vector space V together with a reflexive and transitive graph with vertex set I_n determines a d-ring.

Proof. Let $V = V_1 \oplus V_2 \oplus \ldots \oplus V_n$ be such a decomposition and let G be a reflexive and transitive graph on I_n. Define $J_i = \{j \mid (i,j) \in G\}$. Since G is reflexive, $i \in J_i$ for each $i \in I_n$. Now let

$$\mathfrak{A} = \{\sigma \in \mathcal{L}(V) \mid V_i\sigma \subset \sum_{j \in J_i} \oplus V_j \text{ for each } i \in I_n\}.$$

To show that for $\sigma, \tau \in \mathfrak{A}$, $V_i\sigma\tau \subset \sum_{j \in J_i} \oplus V_j$ for $i \in I_n$, we observe that, since G is transitive, $(i,j) \in G$ implies $J_j \subset J_i$. Hence $V_j\tau \subset \sum_{k\in J_j} \oplus V_k \subset \sum_{k\in J_i} \oplus V_k$ for each $j \in J_i$ and it follows that $\sigma\tau \in \mathfrak{A}$. That the other ring axioms are satisfied can be easily verified. It is clear that $\mathfrak{A}$ satisfies Definition 2.1.

We now prove several lemmas which will be useful in the following development. Let π_i denote the projection of $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ on the i th summand, and let $\mathfrak{A} = \mathfrak{A}(V_i,J_i,n)$ be a d-ring in the lemmas which follow.

2.3 Lemma. If $k \in J_i$, then for each $\sigma' \in \mathrm{Hom}_D(V_i,V_k)$ $\pi_i\sigma' \in \mathrm{Hom}_D(V,V_k) \cap \mathfrak{A}$.

Proof. Let $k \in J_i$, $\sigma' \in \mathrm{Hom}_D(V_i,V_k)$ and $\sigma = \pi_i\sigma'$. Clearly, $\sigma \in \mathrm{Hom}_D(V,V_k)$, and since $V_j\sigma = 0$ for $j \neq i$ and $V_i\sigma = V_i\sigma' \subset V_k$, $\sigma \in \mathfrak{A}$.

2.4 Lemma. If $\sigma \in \mathfrak{A}$ and there exists an $x_o \in V_i$ such that $0 \neq x_o\sigma \in V_k$, then $k \in J_i$.

Proof. Suppose $k \notin J_i$. Then $\sigma \in \mathfrak{A}$ implies that $V_i\sigma \subset \sum_{j\in J_i} \oplus V_j$ and hence $0 \neq x_o\sigma \in V_k \cap \sum_{j\in J_i} \oplus V_j = 0$.

2.5 Lemma. $J_j \subset J_i$ if and only if $j \in J_i$.

Proof. Since $j \in J_j$, $J_j \subset J_i$ implies $j \in J_i$. To prove the converse, let $j \in J_i$ and $k \in J_j$. Let χ_i, χ_j and χ_k be bases for V_i, V_j and V_k, respectively, and let $\chi_i \cup \chi_j \cup \chi_k$ be extended to a basis χ for V. Let $x_o \in \chi \cap V_i$, $x_1 \in \chi \cap V_j$ and $x_2 \in \chi \cap V_k$. Define $\sigma' : \chi \cap V_i \to \chi \cap V_j$ by

$x\sigma' = \begin{cases} 0 \text{ if } x \neq x_o \\ x_1 \text{ if } x = x_o \end{cases}$ and $\tau' : \chi \cap V_j \to \chi \cap V_k$ by

$x\tau' = \begin{cases} 0 \text{ if } x \neq x_1 \\ x_2 \text{ if } x = x_1 \end{cases}$. By an elementary theorem of linear algebra σ' and τ' can be extended to σ and τ of $\mathrm{Hom}_D(V_i, V_j)$ and $\mathrm{Hom}_D(V_j, V_k)$, respectively. By Lemma 2.3, $\pi_i\sigma \in \mathrm{Hom}_D(V, V_j) \cap \mathfrak{A}$. Hence $0 \neq \pi_i\sigma\pi_j\tau \in \mathfrak{A}$ and by Lemma 2.4, $k \in J_i$.

2.6 Definition. With each d-ring $\mathfrak{A} = \mathfrak{A}(V_i, J_i, n)$ we associate a graph $G(J_i, n) = \{(k,j) \mid j \in J_k \text{ for some } k \in I_n\}$. ($G(J_i, n)$ is not uniquely determined by $\mathfrak{A}$; however, when there is no possibility of confusion we shall often abuse the notation by writing $G(\mathfrak{A}) = G(J_i, n)$ and calling this the graph of $\mathfrak{A}$).

2.7 Lemma. $G(\mathfrak{A})$ is a reflexive and transitive directed graph.

Proof. Since $i \in J_i$ for each $i \in I_n$, $(i,i) \in G(\mathfrak{A})$ for each $i \in I_n$. If (i,j), $(j,k) \in G(\mathfrak{A})$, then $j \in J_i$ and $k \in J_j$. From Lemma 2.6, it follows that $J_j \subset J_i$, hence $k \in J_i$. Therefore, $(i,k) \in G(\mathfrak{A})$.

Examples.

(1) Let V be a 4-dimensional vector space over a division ring D with a basis e_1, e_2, e_3, e_4. Let $V_i = De_i$ ($i = 1, 2, 3, 4$) and let $J_1 = \{1\}$, $J_2 = \{2\}$, $J_3 = \{1, 2, 3\}$, and $J_4 = \{2, 4\}$. It is clear that $\mathfrak{A} = \mathfrak{A}(V_i, J_i, 4)$ is isomorphic to the ring of all 4×4 matrices whose elements have the form

$$\begin{Vmatrix} x_{11} & 0 & 0 & 0 \\ 0 & x_{22} & 0 & 0 \\ x_{31} & x_{32} & x_{33} & 0 \\ 0 & x_{42} & 0 & x_{44} \end{Vmatrix}$$

where $x_{ij} \in D$. Then one easily sees that

$$G(\mathfrak{A}) = \{(1,1),(2,2),(3,3),(4,4),(3,1),(3,2),(4,2)\}.$$

(2) One easily shows that any d-ring of linear transformations on a 2-dimensional vector space V over a division ring D (where V is decomposed into the direct sum of two 1-dimensional subspaces) is isomorphic to a ring of 2 x 2 matrices over D whose elements have one of the forms

$$\text{(i)} \begin{Vmatrix} x & 0 \\ 0 & y \end{Vmatrix} \quad \text{(ii)} \begin{Vmatrix} x & 0 \\ z & y \end{Vmatrix} \quad \text{(iii)} \begin{Vmatrix} x & z \\ 0 & y \end{Vmatrix} \quad \text{(iv)} \begin{Vmatrix} x & z \\ w & y \end{Vmatrix}.$$

(3) Let V be an n-dimensional vector space with a basis $\mathfrak{X} = \{v_1, v_2, \ldots, v_n\}$ over a division ring D. Let $V = V_1 \oplus V_2 \oplus \ldots \oplus V_n$ where $V_i = Dv_i$ (the subspace of all scalar multiples of v_i). Finally, let M_n denote the complete ring of all n x n matrices over D and let MU(n) denote the collection of all matrix units e_{ij} in M_n together with the zero matrix. It is well known that $\mathfrak{L}(V) \cong M_n$ and it is clear that $\mathfrak{L}(V)$ is a d-ring with $J_i = I_n$ ($i = 1, 2, \ldots, n$). Moreover, if S is a sub-semigroup of MU(n) then the set $\mathfrak{A}(S,D)$ of all linear combinations over D of elements of S is a d-ring. In this ring $J_i = \{j | e_{ij} \in S\}$, and $G(\mathfrak{A}(S,D)) = \{(i,j) | e_{ij} \in S\}$.

(4) Let V be a vector space of any dimension over a division ring D, and let $V = V_1 \oplus V_2 \oplus \ldots \oplus V_n$ be

any decomposition of V into subspaces V_i. Let G be a reflexive and transitive graph on I_n, let $J_i = \{j \mid (i,j) \in G\}$, and let $M_i = \sum_{j \in X} \oplus V_j$ $(i=1, 2, \ldots, n)$ where $X = \bigcup_{k \in S} J_k$ with $S \subset I_n$. Then the ring of all linear transformations leaving M_i invariant for each $i \in I_n$ is a d-ring. We will say more about this d-ring later.

Before stating Johnson's definition of a distinguished ring [2], we recall some lattice theoretic definitions. A finite sequence $P_o, P_1, \ldots, P_n$ of members of a lattice L is called a chain if $P_i \leq P_{i+1}$ $(i=0, 1, 2, \ldots, n-1)$ and is said to have length n. A chain $P_o, P_1, \ldots, P_n$ is said to connect P and Q of L if $P_o = P$ and $P_n = Q$. An element P of L is said to have lattice dimension n if the longest chain connecting 0 and P has length n, and if L has a maximal element Q and Q has lattice dimension n (n finite), then L is said to be finite dimensional and to have dimension n. Finally, if L is a finite dimensional modular lattice, then an element P in L is called irreducible if $P \neq 0$ (0 is the minimal element of L) and P is not the join of lower dimensional elements of L. Each irreducible $P \in L$ covers a unique element of L which will be designated by P^o.

Two lattices which will be of interest are the lattice of all subspaces of a vector space V and, for a given ring $\mathfrak{A}$ of linear transformations on a vector space V, the lattice of all $\mathfrak{A}$-submodules of V. The join of two elements

P and Q in either of the two lattices is just their sum $P + Q$ and will be denoted by $P \cup Q$. The meet is, of course, the set theoretical intersection.

In [2], Johnson associates with each sublattice J of the lattice $L({}_DV)$ (here ${}_DV$ indicates that V is regarded as a left D-module) of all subspaces of V, a subring $\mathfrak{A}(J) = \{\sigma \in L(V) | N\sigma \subset N \text{ for every } N \in J\}$ of $\mathfrak{L}(v)$. It is clear that J is contained in the lattice $L(V_{\mathfrak{A}})$ (here $V = V_{\mathfrak{A}}$ is regarded as a right $\mathfrak{A}$-module) of all $\mathfrak{A}$-submodules of V. He calls a sublattice J of $L({}_DV)$ a distinquished lattice (d-lattice) if and only if $J = L(V_{\mathfrak{A}})$ for some subring $\mathfrak{A}$ of $\mathfrak{L}(V)$, and a subring $\mathfrak{A}$ of $\mathfrak{L}(V)$ a distinquished ring (d-ring) if and only if $\mathfrak{A} = \mathfrak{A}(J)$ where $J = L(V_{\mathfrak{A}})$. For convenience we call such a ring a d'-ring.

A d'-ring is not necessarily a d-ring. For example, let D be the field of integers mod(3) and let $V = D^3$. Let e_1, e_2, e_3 be the canonical basis and let J be the lattice consisting of the subspaces 0, De_1, De_3, $D(e_1 + e_3)$ $De_1 + De_3$ and V. J is not distributive, since $De_1 \cap (De_3 \cup D(e_1 + e_3)) = De_1$, but $De_1 \cap De_3 = 0 = De_1 \cap D(e_1 + e_3)$. The associated d'-ring $\mathfrak{A}(J)$ is easily seen to be isomorphic to the ring of all 3 x 3 matrices of the form

$$\begin{Vmatrix} x & 0 & 0 \\ y & z & w \\ 0 & 0 & x \end{Vmatrix} \tag{1}$$

where $x, y, z, w \in D$.

It is also true that a d-lattice is not necessarily distributive, even if it is finite. For example, let D be the field of integers modulo 2, let $V = D^2$, and let $\mathfrak{A} = \{\alpha I | \alpha \epsilon D\}$ where

$$I = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} .$$

Then every subspace of V is invariant under $\mathfrak{A}$, and since V is finite, $L({}_D V) = L(V_{\mathfrak{A}})$. But $L({}_D V)$ is not distributive. To see this, let $e_1 = (1,0)$, $e_2 = (0,1)$, $V_1 = De_1$, $V_2 = De_2$ and $V_3 = D(e_1 + e_2)$. Now $V_3 \cap (V_2 \cup V_1) = V$, but $(V_3 \cap V_1) \cup (V_3 \cap V_1) = 0$. Similar examples can be constructed letting D be the integers modulo p (p a prime number), $V = D^n$ where $n \geq 2$ and $\mathfrak{A} = \{\alpha I | \alpha \epsilon D\}$ where I is the identity n x n matrix over D.

However, we can show that if the lattice J is finite and distributive, then the d'-ring is also a d-ring.

In the proof of the lemma which follows we need the definition of a simple idempotent. After Johnson, an idempotent $e \neq 0$ of $\mathfrak{A}(J)$ is called simple if and only if there exists an irreducible element P of J such that $Ve \subset P$ and $Ve \cap P^o = 0$.

2.8 Theorem. If $\mathfrak{A} = \mathfrak{A}(J)$ is a d'-ring and J is finite and distributive, then $\mathfrak{A}$ is a d-ring.

For use in the proof of 2.8, we state without proof Theorem 1.1 of [2].

Theorem. (Johnson) Let L be a complete, complemented, modular lattice with identities 0 and I and J be a finite-dimensional sublattice of L containing 0 and I. For each irreducible $P \in J$, let us select $\overline{P} \in L$ so that $P = \overline{P} \oplus P^o$. Then each non-zero $K \in J$ may be represented in the form $K = \sum_{i=1}^{m} \oplus \overline{P}_i$ for some subset $\{P_1, P_2, \ldots, P_m\}$ of $I(K)$ ($I(K)$ is the set of irreducible elements of J which are contained in K). If J is distributive, then $K = \sum_{i=1}^{n} \oplus \overline{P}_i$ where $I(K) = \{P_1, P_2, \ldots, P_n\}$.

Proof of 2.8. Now to apply this to the proof of the above theorem we let L be the lattice $L({}_D V)$ of all subspaces of the underlying vector space V and let J be the finite, distributive lattice in the statement of the theorem. Let $\{P_1, \ldots, P_n\}$ be the set of all irreducible elements of J. Choose $\overline{P}_i$ in L so that $\overline{P}_i \oplus P_i^o = P_i$. Now since J is distributive and $V \in J$, we have $V = \overline{P}_1 \oplus \ldots \oplus \overline{P}_n$. Let e_i be the projection of V onto $\overline{P}_i$, $i \in I_n$. Then $1 = e_1 + e_2 + \ldots + e_n$ and $e_i e_j = \delta_{ij} e_j$. By the definition of e_i, $Ve_i = P_i \subset \overline{P}_i$ and $Ve_i \cap P_i^o = 0$. Therefore, each e_i is simple. Let G be the graph on I_n defined by

$$G = \{(i,j) \mid e_i \mathfrak{A} e_j \neq 0\}.$$

Since $e_i \mathfrak{A} e_i \neq 0$, for each $i \in I_n$, G is reflexive. By [2, 6.4], $e_i \mathfrak{A} e_j \neq 0$ if and only if $P_j \subset P_i$. Hence, if $e_i \mathfrak{A} e_j$ and $e_j \mathfrak{A} e_k$ are both non-zero, so is $e_i \mathfrak{A} e_k$. Therefore, G is transitive.

By Lemma 2.2, G together with the decomposition $V = Ve_1 \oplus \ldots \oplus Ve_n$ determines a d-ring $\mathfrak{A}' = \mathfrak{A}(Ve_i, J_i, n)$ where $J_i = \{j \mid e_i \mathfrak{A} e_j \neq 0\}$. We need to show that $\mathfrak{A}' = \mathfrak{A}$.

Let $\sigma \in \mathfrak{A}'$. To show that $\sigma \in \mathfrak{A}$, we first prove that $P_i = \sum_{j\in J_i} \oplus Ve_j$. Again, by the theorem of Johnson stated above,

$$P_i = \sum_{j\in S} \oplus Ve_j = \sum_{j\in S} \oplus \overline{P}_j \text{ where } I(P_i) = \{P_j \mid j\in S\}.$$

By [2, 6.4], $P_j \subset P_i$ if and only if $e_1\mathfrak{A}e_j \neq 0$. Therefore, $S = J_i$. Now, since $(\sum_{j\in J_i} \oplus Ve_j)\sigma \subset \sum_{j\in J_i} \oplus Ve_j$, by Lemma 2.5, $P_i\sigma \subset P_i$ for each $i \in I_n$. It follows that $\sigma \in \mathfrak{A}$. To obtain the opposite inclusion, let $\sigma \in \mathfrak{A}$, then

$$\sigma = \sum_{i,j}^{n} e_i\sigma e_j.$$

Note that $Ve_ie_j\mathfrak{A}e_k = 0$ if $j\neq i$ or $e_j\mathfrak{A}e_k = 0$, and $Ve_ie_i\sigma e_j \subset Ve_j$ for each $\sigma \in \mathfrak{A}$. Therefore, since $e_i\sigma e_j \neq 0$ implies that $j \in J_i$, we have that

$$Ve_i\sigma \subset \sum_{j\in J_i} \oplus Ve_j$$

for each $i \in I_n$, i.e., $\sigma \in \mathfrak{A}'$. Therefore, $\mathfrak{A} = \mathfrak{A}'$ and the theorem is proved.

2.9 Corollary. Let $\mathfrak{A} = \mathfrak{A}(J)$ be a d'-ring and let J be finite. If the underlying division ring D has characteristic zero, then $\mathfrak{A}$ is a d-ring.

Proof. In [2, 2.6], Johnson proved that if D has characteristic zero, then every finite d-lattice is distributive.

With each d-ring we now associate a lattice and show that each d-ring is a d'-ring. To accomplish this we first consider an arbitrary, reflexive, transitive graph on I_n. Let $[i] = \{j \mid (i,j) \in G\}$ for each $i \in I_n$. Let $L(G)$ be the set of all finite sums $[i_1] \cup [i_2] \cup \dots \cup [i_k]$ together with the empty set $\emptyset$.

2.10 Lemma. $L(G)$ is a sublattice of $\mathcal{P}(I_n)$ (the lattice of all subsets of I_n) and $[i]$, $i \in I_n$, are the irreducible elements of $L(G)$.

Proof. If $h \in [i] \cap [j]$ and $(h,t) \in G$, then (i,t), $(j,t) \in G$, by transitivity, which by definition implies $t \in [i] \cap [j]$. Therefore $[h] \subset [i] \cap [j]$. Since $\#([i] \cap [j])$ is finite, $[i] \cap [j] = [i_1] \cup [i_2] \cup \dots \cup [i_k]$.

If $[i]$ were irreducible, then $[i] = [h_i] \cup [h_2] \cup \dots \cup [h_k]$ where $[h_j]$ is of lower dimension than $[i]$ for $(j = 1, 2, \dots, k)$. Now $i \in [h_t]$ implies $(h_t, i) \in G$ and applying transitivity, we obtain $[i] \subset [h_t]$. Hence $[i] = [h_t]$, a contradiction. Therefore, $[i]$ is irreducible.

We associate with each d-ring $\mathfrak{A} = \mathfrak{A}(V_i, J_i, n)$ the lattice $L(\mathfrak{A}) = L(\mathfrak{A}(V_i, J_i, n)) = L(G(\mathfrak{A}))$ (see 2.6). Note that $[i] = J_i$ for each $i \in I_n$. $L(\mathfrak{A})$ is distributive since it is a sublattice of the lattice of all subsets of I_n. Let $\mathcal{J}(\mathfrak{A}) = \{M \mid M = \sum_{j \in X} \oplus V_j \text{ for some } X \in L(\mathfrak{A})\} \cup \{0\}$. One easily verifies that $\mathcal{J}(\mathfrak{A})$ is a lattice under intersection and sum and that $X \to \sum_{j \in X} \oplus V_j$ is a lattice isomorphism from $L(\mathfrak{A})$ onto $\mathcal{J}(\mathfrak{A})$.

2.11 Theorem. Let $\mathfrak{A} = \mathfrak{A}(V_i, J_i, n)$ be a d-ring, then $\mathfrak{A}(\mathcal{J}(\mathfrak{A})) = \mathfrak{A}$ and $\mathcal{J}(\mathfrak{A}) = L(V_{\mathfrak{A}})$ where $\mathfrak{A}(\mathcal{J}(\mathfrak{A}))$ and $L(V_{\mathfrak{A}})$ are as defined on p. 14, i.e., every d-ring is a d'-ring.

Proof. The proof will be divided into the three lemmas which follow.

2.12 Lemma. Let $M \in \mathcal{J}(\mathfrak{A})$. Then M is an $\mathfrak{A}$-submodule of V.

Proof. Let $M \in \mathcal{J}(\mathfrak{A})$, then $M = \sum_{j \in X} \oplus V_j$. Hence,

$M\sigma \subset \sum_{k \in K} \oplus V_k$ where $K = \bigcup_{j \in X} J_j$. Since $[i] = \mathcal{J}_i$ for each

$i \in I_n$ and $X = [i_1] \cup [i_2] \cup \ldots \cup [i_t]$,

$$K = \bigcup_{j \in X} J_j = \bigcup_{j \in X} [j] = X.$$

Therefore, M is an $\mathfrak{A}$-submodule of V.

2.13 Lemma. If $\sigma \in \mathcal{L}(V)$ such that $M\sigma \subset M$ for each $M \in \mathcal{J}(\mathfrak{A})$, then $\sigma \in \mathfrak{A}$.

Proof. Let $\sigma \in \mathcal{L}(V)$ such that $M\sigma \subset M$ for each $M \in \mathcal{J}(\mathfrak{A})$. Then $V_i\sigma \subset \sum_{j \in [i]} \oplus V_j$ since $M\sigma \subset M$ for

$M = \sum_{j \in [i]} \oplus V_j$. Therefore, $V_i\sigma \subset \sum_{j \in J_i} \oplus V_j$, for each $i \in I_n$,

i.e., $\sigma \in \mathfrak{A}$.

2.14 Lemma. If M is an $\mathfrak{A}$-submodule of V, then $M \in \mathcal{J}(\mathfrak{A})$.

Proof. We first show that if $M \cap V_j \neq 0$ for some $j \in I_n$, then $V_j \subset M$. Suppose not. Then there is an $x_0 \neq 0$ in $V_j \setminus M$. Since $M \cap V_j \neq 0$, there is an element $x_1 \neq 0$ in the intersection. Let $\{x_1\}$ be extended to a basis for V and define a function σ' on the basis by

$$x\sigma' = \begin{cases} 0 \text{ if } x \neq x_1 \\ x_0 \text{ if } x = x_1 . \end{cases}$$

Now σ' can be extended to a linear transformation σ on V such that $V_i\sigma = 0$ if $i \neq j$ and $V_j\sigma \subset V_j$. But $x_1\sigma = x_0 \notin M$ which contradicts the fact that M is an $\mathfrak{A}$-submodule of V. Therefore, $v_j \subset M$.

Now suppose $M \cap V_i = 0$ for all $i \in I_n$ but $M \neq 0$. Since the projection π_i is in $\mathfrak{A}$ for each $i \in I_n$, $M\pi_j = 0$ for some $j \in I_n$. By the argument in the paragraph above, $M\pi_j = V_j$. Let $K = \{i | M\pi_i = 0\}$. Now it is clear that

$$M \subset \sum_{i \in K} M\pi_i = \sum_{i \in K} V_i \subset M.$$

It suffices to show that $K = \bigcup_{j \in K} [j]$. This follows if we show that $i \in K$ implies that $[i] \subset K$. But this is clear since $[i] = J_i$ and $\mathfrak{A}$ contains all $\sigma \in \mathfrak{L}(V)$ such that

$$V_i\sigma \subset \sum_{j \in J_i} \oplus V_j .$$

This completes the proof of Theorem 2.10.

The reader who is unfamiliar with concepts of this chapter will find a more than adequate background in [9], [10] and [12].

CHAPTER III

A NORMAL FORM FOR DISTINGUISHED RINGS

Let $\mathfrak{A} = \mathfrak{A}(V_i,J_i,n)$ be a d-ring of linear transformations on V. If a basis χ for V is taken such that $\chi = \bigcup_{i=1}^{n} \chi_i$ where χ_i is a basis for V_i, then the matrix of an element of $\mathfrak{A}$ with respect to this basis will be of the form

$$\begin{Vmatrix} X_{11} & \cdots & X_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ X_{n1} & \cdots & X_{nn} \end{Vmatrix}$$

where $X_{ij} = 0$ if $j \notin J_i$ (equivalently $X_{ij} = 0$ if (i,j) $(i,j) \notin G(\mathfrak{A}) = G(J_i,n)$). Moreover, any block matrix $Y = (Y_{ij})$ for which $Y_{ij} = 0$ if $j \notin J_i$ (or $(k,j) \notin G(\mathfrak{A})$) is the matrix of some element of $\mathfrak{A}$ with respect to the basis χ. The purpose of this chapter is to show that for a given d-ring $\mathfrak{A}(V_i,J_i,n)$ we may choose a new triple (V_i',J_i',n') such that $\mathfrak{A}(V_i,J_i,n) = \mathfrak{A}(V_i',J_i',n')$ and such that the graph G' of $\mathfrak{A}$ with respect to the new "coordinate system" (V_i',J_i',n') is in triangular form, i.e., $(i,j) \in G$ implies $j \leq i$. This implies that if we adapt a basis χ' of V to the decomposition $V = \sum_{i=1}^{n'} \oplus V_i'$, then the matrix of each element of $\mathfrak{A}$ with respect to the new basis χ' has the form

$$\left\| \begin{array}{ccccc} X_{11} & & & \bigcirc & \\ X_{21} & X_{22} & & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & & \cdot & \\ \cdot & \cdot & & & \cdot \\ X_{n'1} & X_{n'2} & \cdot & \cdot \; \cdot \; \cdot & X_{n'n'} \end{array} \right\|$$

where $X_{ij} = 0$ if $(i,j) \notin G$.

The following example may be helpful.

Example. Let V be any vector space of $\dim \geq 3$ and let $V = V_1 \oplus V_2 \oplus V_3$ where $V_i \neq 0$. Let $J_1 = \{1,3\}$, $J_2 = \{2,2\}$, $J_3 = \{1,3\}$. Then $\mathfrak{A}(V_i, J_i, 3)$ is isomorphic to the ring of all block matrices of the form

$$\left\| \begin{array}{ccc} A_{11} & 0 & A_{13} \\ 0 & A_{22} & 0 \\ A_{31} & 0 & A_{33} \end{array} \right\|$$

where A_{ij} is a $(\dim V_i) \times (\dim V_j)$ row finite matrix. Now let $V_1' = V_1$, $V_2' = V_3$, $V_3' = V_2$ and let $J_1' = \{1,2\}$, $J_2' = \{1,2\}$, $J_3' = \{3\}$. Then one easily verifies that $\mathfrak{A}(V_i, J_i, 3) = \mathfrak{A}(V_i', J_i', 3)$ and that if we adapt a basis to the V_i', then $\mathfrak{A}$ "takes" the form:

$$\left\| \begin{array}{ccc} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{array} \right\|$$

It is now clear that this may be simplified further by choosing $V'' = V_1' \oplus V_2'$, $V_2'' = V_3'$, $J_1'' = \{1\}$ and $J_2'' = \{2\}$.

We still have $\mathfrak{A}(V_i'', J_i'', 2) = \mathfrak{A}(V_i, J_i, 3)$ and, moreover, in a basis adapted to this new decomposition $\mathfrak{A}$ "takes" the form

$$\begin{Vmatrix} A_{11} & 0 \\ 0 & A_{22} \end{Vmatrix}$$

where A_{ii} is a $(\dim V_i'') \times (\dim V_i'')$ row finite matrix.

Let $\mathfrak{A} = \mathfrak{A}(V_i, J_i, n)$ be a d-ring, let π be a permutation on I_n and let $k_i = i\pi^{-1}$. For each $i \in I_n$, define $(J_i, \pi) = \{k\pi \mid k \in J_{k_i}\}$. Then V may be expressed as $V = (V_1, \pi) \oplus (V_2, \pi) \oplus \ldots \oplus (V_n, \pi)$ where $(V_i, \pi) = V_{k_i}$.

3.1 Lemma. $\mathfrak{A}(V_i, J_i, n) = \mathfrak{A}((V_i, \pi), (J_i, \pi), n)$.

Proof. For each $i \in I_n$, $k_i\pi = i$, and since $k_i \in J_{k_i}$, $i = k_i\pi \in (J_i, \pi)$. Now $k_j \in J_{k_i}$ if and only if $j \in (J_i, \pi)$; hence $\sum_{k_j \in J_{k_i}} \oplus V_{k_j} = \sum_{j \in (J_i, \pi)} \oplus (V_j, \pi)$ for each $i \in I_n$. It follows that $V_{k_i}\sigma \subset \sum_{k_j \in J_{k_i}} \oplus V_{k_j}$ for each $i \in I_n$ if and only if $(V_i, \pi)\sigma \subset \sum_{j \in (J_i, \pi)} \oplus (V_j, \pi)$ for each $i \in I_n$. Therefore, $\mathfrak{A}((V_i, \pi), (J_i, \pi), n) = \mathfrak{A}$.

Since, for a given permutation π on I_n, $\mathfrak{A}(V_i, J_i, n) = \mathfrak{A}((V_i, \pi), (J_i, \pi), n)$, it is obvious that the corresponding matrix rings are isomorphic. We note that this procedure essentially effects a change of basis.

3.2 Definition. Let $\mathfrak{A} = \mathfrak{A}(V_i, J_i, n)$ be a d-ring. The associated graph $G(\mathfrak{A}) = G(J_i, n)$ is said to be in normal form if and only if there exists a set $\{P_1, P_2, \ldots, P_m\} \subset I_n$ such that $n = \sum_{i=1}^{m} P_i$, and for $q_k = \sum_{i=1}^{k} P_i$ $(k = 1, 2, \ldots, m)$, $q_o = 0$, we have

(i) $s \in I_m \cup \{0\}$ and $q_{s-1} < i \leq q_s$ implies $(i,j) \notin G(\mathfrak{A})$ for each $j > q_s$, and

(ii) if $s,t \in I_m \cup \{0\}$, $q_{s-1} < i_o \leq q_s$ and $(i_o,j_o) \in G(\mathfrak{A})$ for some j_o such that $q_{t-1} < j_o \leq q_t$, then $(i,j) \in G(\mathfrak{A})$ for each i and each j such that $q_{s-1} < i \leq q_s$, $q_{t-1} < j \leq q_t$.

It follows that if $\mathfrak{A} = \mathfrak{A}(V_i,J_i,n)$ is a d-ring and $G(\mathfrak{A})$ is in normal form, then (with respect to a basis adapted to the V_i) each element X of $\mathfrak{A}$ has the form

$$X = \left\| \begin{array}{cccc} X_{11} & & & \\ X_{21} & X_{22} & & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ X_{m1} & X_{m2} & \cdots & X_{mm} \end{array} \right\| \quad (2)$$

where X_{ij} may be chosen arbitrarily if $(i,j) \in G(\mathfrak{A})$ and $X_{ij} = 0$ if $(i,j) \notin G(\mathfrak{A})$.

The problem of putting the graph of a d-ring $\mathfrak{A} = \mathfrak{A}(V_i,J_i,n)$ in normal form can be solved using a graph theoretical argument. Let $G = G(J_i,n)$ and let $G = G((J_i,\pi),n)$. One easily shows that $G_\pi = \{(i\pi,i\pi) | (i,j) \in G\}$. This reduces the problem to finding a permutation π such that G_π has normal form. In [1], Clark deduces from the results of Gauntmacher [11] and Dulmage and Mendelsohn [8] that this can always be done for any transitive, finite directed graph.

In [4], Gluskin obtains a normal form for rings of this general class where D is a Euclidean ring. For our purposes, however, the result of Gluskin as well as that of Dulmage and Mendelsohn is far more than what is needed. We therefore include the following theorem and proof.

3.3 Theorem. For each d-ring $\mathfrak{A}(V_i,J_i,n)$ there is a permutation π on I_n such that $G((J_i,\pi),n)$ is in normal form.

Proof. We proceed by induction on n. If $n = 1$, then $\mathfrak{A} = \mathfrak{L}(V)$ and $G(\mathfrak{A})$ is clearly in normal form. Hence the identity permutation suffices. Suppose the theorem is true for all $n > p$ and consider $\mathfrak{A}(V_i,J_i,n=p)$. Order $\{J_1,J_2,\ldots,J_n\}$ by $J_i \le J_j$ if and only if $\#J_i \le \#J_j$ and let J_a be a minimal element. Let the elements of J_a be ordered according to their numerical values, say $J_a = \{k_1,k_2,\ldots,k_n\}$ where $k_i < k_{i+1}$ $(i=1,2,\ldots,r-1)$. Also, order the elements of $I_n \backslash J_a$ according to their numerical values to obtain $I_n \backslash J_a = \{k_{r+1},k_{r+2},\ldots,k_{r+c=n}\}$ where $k_i < k_{i+1}$ $(i=r+1,r+2,\ldots,(r+c)-1=n-1)$. Define the permutation π on I_n by $i\pi^{-1} = k_i$. Since $a \in J_a$, $a = k_b$ for some b such that $1 \le b \le r$. Accordingly, $(\pi_1,J_b) = \{1,2,\ldots,r\}$. We now show that $(\pi_1,J_b) = (\pi_1,J_i)$ for each $i \le r$. If $k_i \in J_a$, then $J_{k_i} \subset J_a$, by Lemma 2.5. Since $a = k_b$ and J_a is minimal, $J_a = J_{k_b} = J_{k_i}$. Therefore

$(\pi_1, J_b) = (\pi_1, J_i)$ for each $i \le r$.

Now let $V' = V_{r+1} \oplus V_{r+2} \oplus \dots \oplus V_{r+c=n}$. Then $\mathfrak{A}'(V_{r+i}, (\pi_1, J_{r+i})', c=n-r)$ is a d-ring with $c < p$ where $(\pi_1, J_{r+i})' = (\pi_1, J_{r+i})\, I_r \backslash (I_r = \{1,2,\dots,r\})$. By the inductive hypothesis, there is a permutation π' on $I_c = \{1,2,\dots,c=m-r\}$ such that $G(\mathfrak{A}')$ is in normal form. Define the permutation π_2 on I_n by

$$i\pi_2 = \begin{cases} i \text{ if } i \le r \\ r+j\pi' \text{ if } j \le c=n-r, \end{cases}$$

and let $\pi = \pi_1\pi_2$. Thus

$$k_i\pi = \begin{cases} i \text{ if } i \le r \\ r+j\pi' \text{ if } j \le c=n-r. \end{cases}$$

We claim that $G((J_i,\pi),n)$ is in normal form. From the definition of π, it follows that

$(J_{r+j},\pi)\backslash I_r = (J_{r+j},\pi')$ for all $j \in I_c$. Hence, for $i > r$, $j > r$

$$G((J_i,\pi),n) = G(J_{r+j},\pi'). \qquad (3)$$

Since $G(J_{r+j},\pi')$ is in normal form, there is a set $\{p_1', p_2', \dots, p_{t=m-1}'\} \subset I_n$ such that $c = m-r$

$$c = n-r = \sum_{i=1}^{m-1} p_i' \text{ and for } q_k = \sum_{i=1}^{k} p_i' \; (k=1,2,\dots,n-r)$$

(i) and (ii) of the definition of normal form are satisfied. Let $p_{i+1} = p_i'$ $(i=1,2,\dots,m-1)$ and let $p_1 = r$. Then

$$n = \sum_{i=1}^{m} p_i \text{ and for } q_k = \sum_{i=1}^{k} p_i \; (k=1,2,\dots,m)$$ (i) of the

definition of normal form is satisfied, since (3) holds and $\pi = \pi_1$ on $\{k_1,k_2,\ldots,k_r\}$ implies $(J_i,\pi) = (J_i,\pi_1)$ for all $i \in I_r$. Again, by (3) and the fact that $(J_i,\pi) = (J_i,\pi_1)$ for all $i \in I_r$, we need only prove:

> If $q_{s-1} < i_o \le q_s$, $0 = q_o < j_o \le q_1$ and $(i_o,j_o) \in G((J_i,\pi),n)$, then $(i,j) \in G((J_i,\pi),n)$ for each i and each j such that $q_{s-1} < i \le q_s$, $0 = q_o < j \le q_1$.

Let the hypothesis of this statement be satisfied and let $0 < j \le q_1$. Since $j_o \in (J_j,\pi)$, $(J_o,j) \in G((J_i,\pi),n)$ and by transitivity $(i_o,j) \in G((J_i,\pi),n)$. By (3), $(i,i_o) \in G((J_i,\pi),n)$ for each i such that $q_{s-1} < i < q_s$. Again, by transitivity, $(i,j_o) \in G((J_i,\pi),n)$. One more application of transitivity shows $(i,j) \in G((J_i,\pi),n)$ for each i and each j such that $0 < j \le q_1$ and $q_{s-1} < i < q_s$, and the theorem is proved.

We will have occasion in the next chapter to partition further the elements of a d-ring $\mathfrak{A} = \mathfrak{A}(V_i,J_i,m)$ whose graph $G(\mathfrak{A})$ is in normal form. The elements of $\mathfrak{A}$ with respect to a basis adapted to the V_i have the form

$$\left\| \begin{array}{cccc|c} X_{11} & 0 & 0 & \cdots & 0 \\ X_{21} & X_{22} & 0 & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ X_{m-1\ 1} & X_{m-1\ 2} & \cdots & X_{m-1\ m-1} & 0 \\ \hline X_{m\ 1} & X_{m\ 2} & \cdots & X_{m\ m-1} & X_{m\ m} \end{array} \right\| \qquad (4)$$

We partition the elements of $\mathfrak{A}$ along the dotted lines of (4).

Elements of $\mathfrak{A}$ may then be expressed in the form

$$\left\| \begin{matrix} B_{11} & 0 \\ B_{21} & B_{22} \end{matrix} \right\| \qquad (5)$$

where B_{11} is the $m-1 \times m-1$ block matrix in the upper lefthand corner of (4), and B_{21} and B_{22} are determined accordingly. Since we have assumed that $G(\mathfrak{A})$ is in normal form, there is a set $\{p_1,p_2,\ldots,p_n\} \subset I_n$ such that

$$n = \sum_{i=1}^{n} p_i .$$

Let $q_k = \sum_{i=1}^{k} p_i$. Then $V' = V_1 \oplus V_2 \oplus \ldots \oplus V_{q_{m-1}}$ together with $J_i' = J_i \backslash \{q_{m-1}+1, q_{m-1}+2,\ldots,q_m\}$ $(i=1,2,\ldots,q_{m-1})$ determine a d-ring $\mathfrak{A}'(V_i,J_i',q_{m-1}) \subset \mathfrak{L}(V')$. Considering the partitioning of elements of $\mathfrak{A}$ as in (4) and the alternative representation in (5), it is clear that each B_{11} of (5) is the matrix of an element of $\mathfrak{A}'(V_i,J_i',q_{m-1})$. We will write $\mathfrak{B}_{11}$ for $\mathfrak{A}'(V_i,J_i',q_{m-1})$ and let $\mathfrak{B}_{22}$ denote the full ring of blocks X_{mm}. $\mathfrak{B}_{21}$ will denote the set of matrices

$$\left\| X_{m\,1},\; X_{m\,2},\; \ldots,\; X_{m\,m-1} \right\|$$

occurring in the lower lefthand corner of (4).

CHAPTER IV

RICKART AND BAER RINGS

It is known [5] that if V is a vector space and $0 \subset V_1 \subset V_2 \subset V_3 \ldots \subset V_n = V$ is a chain of subspaces, then the ring of all linear transformations leaving each element of the chain invariant is Baer, hence Rickart (see definitions below). In this chapter we show that any d-ring $\mathfrak{A}$ of linear transformations on V is a Rickart ring if and only if the graph G (with respect to some decomposition of V) of $\mathfrak{A}$ surrounds no zero (defined below). We also show, using a result of Johnson [2], that a d-ring $\mathfrak{A}$ is non-Baer if the graph G of $\mathfrak{A}$ surrounds a zero.

Let $\mathfrak{A}$ be a ring and S be a subset of $\mathfrak{A}$. The right (respectively, left) annihilator of S, denoted $r_{\mathfrak{A}}(S)$ (respectively, $l_{\mathfrak{A}}(S)$), is the set of all $\sigma \epsilon \mathfrak{A}$ such that $\tau\sigma = 0$ (respectively $\sigma\tau = 0$) for all τ in S. After Kaplansky [3], we call a ring $\mathfrak{A}$ Baer if the right (respectively left) annihilator of every subset of $\mathfrak{A}$ is a principal right (respectively left) ideal generated by an idempotent. As in [6], $\mathfrak{A}$ is called a Rickart ring if the right (respectively left) annihilator of any element of $\mathfrak{A}$ is a principal right (respectively left) ideal generated by an idempotent.

If G is a graph with a vertex set I_n, and if $J \subset I_n$, then $G' = G \cap (J \times J) = \{(i,j) \in G \mid (i,j) \in J \times J\}$ is called a *subgraph of* G (*with respect to the vertex set* J). A k-*section* of G is a subgraph of G with a vertex set J of cardinality k.

4.1 *Definition*. A graph G is said to *surround a zero* if and only if G contains a 4-section

$G' = \{(h,h),(i,i),(j,j),(k,k),(i,h),(j,h),(k,h),(k,i),(k,j)\}$.

Now, if G is the graph of a d-ring $\mathfrak{A} = \mathfrak{A}(V_i, J_i, n)$, then G surrounds a zero if and only if there is an idempotent e in $\mathfrak{A}$ such that each element of the matrix representation of $e\mathfrak{A}e$ has the form

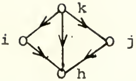
$$\begin{Vmatrix} X_{11} & 0 & 0 & 0 \\ X_{21} & X_{22} & 0 & 0 \\ X_{31} & 0 & X_{33} & 0 \\ X_{41} & X_{42} & X_{43} & X_{44} \end{Vmatrix}$$

with an appropriate number of zero block rows and block columns interpolated (see Lemma 4.4 and its proof).

One can easily verify that a graph G surrounds no zero if and only if it has the following property:

> If h, i, j, and k are distinct integers such that $(k,i),(k,j),(i,h),(j,h) \in G$, then either (i,j) or $(j,i) \in G$.

In graph theoretical terminology this means that G has no subgraph isomorphic to the following graph:



We note that if $\mathfrak{A} = \mathfrak{A}(V_i,J_i,n)$ is a d-ring and π is a permutation on I_n, then $G(J_i,n)$ surrounds a zero if and only if $G((J_i,\pi),n)$ does. Hence, the property of surrounding a zero is invariant under permutations of the coordinate system (V_i,J_i,n) of a d-ring.

We now make a definition which will be needed in the lemmas which follow:

4.2 Definition. A graph G on I_n is said to be triangular if and only if $(i,j) \in G$ implies $j \leq i$.

Now, if $\mathfrak{A} = \mathfrak{A}(V_i,J_i,n)$ is a d-ring whose graph G is in normal form, then a suitable coordinate system (V_i',J_i',n) can be chosen such that $\mathfrak{A} = \mathfrak{A}(V_i',J_i',n)$ and $G' = G'(J_i',n)$ is triangular. This can be accomplished in the following way: Recall from the definition of normal form 3.2 that there exists a set $\{p_1,p_2,\ldots,p_n\} \subset I_n$ such that

$$n = \sum_{i=1}^{m} p_i \quad \text{and for} \quad q_k = \sum_{i=1}^{k} p_i \ (k = 1,\ 2,\ \ldots,\ m)$$

(i) and (ii) are satisfied. Let

$$V_i' = \sum_{j=q_{i-1}}^{q_i} \oplus\, V_j \ (i = 1,\ 2,\ \ldots,\ m),$$

and define the graph G' on I_m by $(i,j) \in G'$ if and only if $q_{i-1} < h \leq q_i$ and $q_{j-1} < k \leq q_j$ implies $(h,k) \in G$. It follows that $\mathfrak{A} = \mathfrak{A}(V_i',J_i',m)$, where $J_i = \{j \mid (i,j) \in G'\}$, and that $G' = G'(J_i',m)$ is triangular. One easily verifies that G' surrounds a zero if and only if G does.

Note that a reflexive and transitive graph G is a partial order if G is triangular.

Recall that in Chapter III we stated that we would have occasion to partition the elements of a d-ring $\mathfrak{A} = \mathfrak{A}(V_i,J_i,n)$ as in (5). In view of the fact that for any d-ring we may choose a new coordinate system (V_i',J_i',m) such that $G(J_i',m)$ is triangular, (5) becomes

$$\left\| \begin{array}{cccc|c} X_{11} & 0 & & & 0 \\ X_{21} & X_{22} & & & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ X_{m-1\,1} & X_{m-1\,2} & \cdots & X_{m-1\,m-1} & 0 \\ \hline X_{m1} & X_{m2} & \cdots & X_{m\,m-1} & X_{m\,m} \end{array} \right\| \tag{6}$$

The elements of $\mathfrak{A} = \mathfrak{A}(V_i',J_i',m)$ may now be expressed in the form

$$\left\| \begin{array}{cc} B_{11} & 0 \\ B_{21} & B_{22} \end{array} \right\| , \tag{7}$$

where B_{11} is the $m-1 \times m-1$ block matrix in the upper left hand corner of (6) and B_{22} and B_{21} are determined accordingly. We shall let $\mathfrak{B}_{ij}$ denote the set of all matrices B_{ij} which appear as blocks in the above matrix representation of $\mathfrak{A}$. As previously pointed out, $\mathfrak{B}_{11}$ is the matrix representation of the d-ring obtained by restricting $\mathfrak{A}$ to $V_1' \oplus \dots \oplus V_{m-1}'$.

We now embark upon the proof of our main result, namely, that a d-ring $\mathfrak{A}(V_i, J_i, n)$ is Rickart if and only if its graph $G = G(J_i, n)$ surrounds no zeros. By Theorem 3.3 and the above comments, we may assume that G is triangular and we do so in the following lemmas. We shall furthermore identify each element of $\mathfrak{A}$ with its matrix (with respect to a basis adapted to the V_i).

4.3 Lemma. Let $\mathfrak{A}$ be a Rickart ring. Then

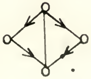
(i) $\mathfrak{A}$ has an identity element, and

(ii) $e\mathfrak{A}e$ is Rickart for each idempotent e in $\mathfrak{A}$.

Proof. (i) $l_{\mathfrak{A}}(0) = \mathfrak{A}e$ for some $e^2 = e \epsilon \mathfrak{A}$ and $r_{\mathfrak{A}}(0) = f\mathfrak{A}$ for some $f^2 = f \epsilon \mathfrak{A}$. Now $\sigma e = \sigma$ for each $\sigma \epsilon \mathfrak{A}$ and $f\sigma = \sigma$ for each $\sigma \epsilon \mathfrak{A}$. Therefore, $e = fe = f$.

(ii) Let $e^2 = e \epsilon \mathfrak{A}$ and $x \epsilon e\mathfrak{A}e$. Now $l_{e\mathfrak{A}e}(x) = e\mathfrak{A}e \cap l_{\mathfrak{A}}(x)$ and $l_{\mathfrak{A}}(x) = \mathfrak{A}f$ for some $f^2 = f \epsilon \mathfrak{A}$. Since e is an identity for $e\mathfrak{A}e$, $(1-e) \subset l_{\mathfrak{A}}(x)$. Hence $(1-e)f = (1-e)$ and $(1-e)fe = 0$. Let $g = efe$. Then $g^2 = g$, since $fe = efe$ implies $fe = fefe$ which in turn implies $efe = efe^2fe$. Clearly, $e\mathfrak{A}e(efe) \subset l_{e\mathfrak{A}e}(x)$. Let $y \epsilon l_{e\mathfrak{A}e}(x)$, then $y \epsilon l_{\mathfrak{A}}(x)$ and for $y = e\sigma e$, $y = yf = e\sigma ef$ and $ye = y$. Hence, $y = yf = yfe = e\sigma efe = e\sigma e^2 fe = yg$. Therefore, $l_{e\mathfrak{A}e}(x) = \mathfrak{A}g$.

4.4 Lemma. Let $\mathfrak{A} = \mathfrak{A}(V_i, J_i, n)$ be a d-ring and let $G = G(J_i, n)$ be triangular. If G surrounds a zero, then there is an idempotent e in $\mathfrak{A}$ such that $e\mathfrak{A}e$ is isomorphic to a d-ring $\mathfrak{A}' = \mathfrak{A}'(V_i', J_i', 4)$ whose graph is of the form



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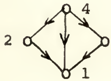
Proof. Since G surrounds a zero, G has a subgraph $G' = \{(h,h),(i,i),(j,j),(k,k),(i,h),(j,h),(k,h),(k,i),(k,j)\}$ on $\{h,i,j,k\} \subset I_n$. Let e_q denote the projection of V = $V = V_1 \oplus \ldots \oplus V_n$ onto V_q and let $e = e_h + e_i + e_j + e_k$. Since e is a projection from V onto $V' = V_h \oplus V_i \oplus V_j \oplus V_k$ we may think of each element of $e\mathfrak{A}e$ as being a linear transformation on V'. One may verify that modulo this identification $e\mathfrak{A}e$ is a d-ring of linear transformations on V' which, with respect to the decomposition $V' = V_h \oplus V_i \oplus V_j \oplus V_k$, has graph G'.

4.4' *Lemma*. If $\mathfrak{A} = \mathfrak{A}(V_i, J_i, 4)$ is a d-ring whose graph G has the form



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then $\mathfrak{A}$ is not Rickart.

Proof. Suppose $\mathfrak{A}$ were Rickart. Let χ_i be a basis for V_i and $\chi = \bigcup_{i=1}^{4} \chi_i$. Let v_i be a fixed element of χ_i $(i = 1,2,3,4)$. Denote by e_i the projection of $V = \sum \oplus \{Dv \mid v \in \chi\}$ onto $Dv_i \subset V_i$ $(i = 1,2,3,4)$. Let $V' = V_1' \oplus V_2' \oplus V_3' \oplus V_4'$ where $V_i' = Dv_i = Ve_i$ and let $G' = G(J_i, 4)$. By Lemma 2.2, this decomposition ov V' together with G' determines the d-ring $\mathfrak{A}'(V_i', J_i', 4)$ which is easily seen to be isomorphic to the ring of all 4 x 4 matrices of the form

$$\left\| \begin{matrix} x_{11} & 0 & 0 & 0 \\ x_{21} & x_{22} & 0 & 0 \\ x_{31} & 0 & x_{33} & 0 \\ x_{41} & x_{42} & x_{43} & x_{44} \end{matrix} \right\|$$

where $x_{ij} \in D$. Let $e = e_1 + e_2 + e_3 + e_4$. Since $\mathfrak{A}$ is Richart $e\mathfrak{A}e$ is Rickart, by Lemma 4.3. Now $e\mathfrak{A}e = \mathfrak{A}'(V_i', J_i', 4)$, for, clearly, $e\sigma e \in \mathfrak{L}(V')$ for each $\sigma \in \mathfrak{A}$, and since $G' = G$, $V_i'\, e\sigma e \subset \sum_{j \in J_i'} \oplus V_j'$. But $\mathfrak{A}'$ is not Rickart, since the left annihilator of

$$\left\| \begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \right\|$$

is not generated by an idempotent, which contradicts the fact that $e\mathfrak{A}e$ is Rickart. Therefore, $\mathfrak{A}$ is not Rickart.

4.5 *Lemma*. Let $\mathfrak{A} = \mathfrak{A}(V_i, J_i, n)$ be a d-ring whose graph G is triangular. Suppose each element of $\mathfrak{A}$ is partitioned in the form

$$\left\| \begin{matrix} B_{11} & 0 \\ B_{21} & B_{22} \end{matrix} \right\| .$$

Let $\mathfrak{B}_{11}$ be Rickart and let $\mathfrak{B}_{22}$ be the full ring of row finite matrices over D. Assume that $\mathfrak{A}$ has the following property:

$$\left.\begin{array}{l} \text{If } A \in \mathfrak{B}_{11} \text{ and } l_{\mathfrak{B}_{11}}(A) = \mathfrak{B}_{11}E \text{ for some} \\ E^2 = E \in \mathfrak{B}_{11}, \text{ then } Y_{21}A = 0 \text{ implies} \\ Y_{21}E = Y_{21} \text{ for each } Y_{21} \in \mathfrak{B}_{21}. \end{array}\right\} \quad (8)$$

Then $\mathfrak{A}$ is Rickart.

Proof. Let

$$x = \begin{Vmatrix} A & 0 \\ B & C \end{Vmatrix} \in \mathfrak{A}$$

then $l_{\mathfrak{B}_{11}}(A) = \mathfrak{B}_{11}E$ for some $E^2 = E \in \mathfrak{B}_{21}$. Let $\mathfrak{A}^*$ be the ring of all row finite matrices over D of the form

$$\begin{Vmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{Vmatrix}$$

where X_{ij} is any row finite matrix of the same dimension as B_{ij}.

By Wolfson's result [5], $\mathfrak{A}^*$ is Baer.

Let

$$A = \begin{Vmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n-1\,1} & \cdots & A_{n-1\,n-1} \end{Vmatrix}$$

where the A_{ij} come from the partition induced by the V_i. (Actually A is in block triangular form.) The elements of $\mathfrak{B}_{21}$ have the form

$$\begin{Vmatrix} X_{n1} & X_{n2} & \cdots & X_{n\,n-1} \end{Vmatrix},$$

where again the X_{ij} are blocks which arise from the partition induced by the V_i.

Define $A^* = (A^*_{ij})$ where

$$A^*_{ij} = \begin{cases} 0 \text{ if } (n,i) \notin G \\ A_{ij} \text{ if } (n,i) \in G \end{cases}$$

$(i,j = 1,2,\ldots,n-1)$.

Note that $X_{ni} = 0$ if $(n,i) \notin G(\mathfrak{A})$.

Since A^* is Baer and

$$\left\| \begin{matrix} A^* & 0 \\ B & C \end{matrix} \right\| \in \mathfrak{A}^*$$

there exists an idempotent

$$E' = \left\| \begin{matrix} E'_{11} & 0 \\ E'_{21} & E'_{22} \end{matrix} \right\| \text{ in } \mathfrak{A}^* \text{ such that}$$

$$l_{\mathfrak{A}^*} \left(\left\| \begin{matrix} A^* & 0 \\ B & C \end{matrix} \right\| \right) = \mathfrak{A}^* \left\| \begin{matrix} E'_{11} & 0 \\ E'_{21} & E'_{22} \end{matrix} \right\| .$$

Let $E'_{21} = \left\| E_{n1}, E_{n2}, \ldots, E_{n\ n-1} \right\|$ and set

$$E^*_{21} = \left\| E^*_{n1}, E^*_{n2}, \ldots, E^*_{n\ n-1} \right\| \text{ where}$$

$$E^*_{ni} = \begin{cases} E_{ni} \text{ if } (n,i) \in G \\ 0 \text{ if } (n,i) \notin G. \end{cases}$$

(It may happen that $E^*_{21} = E'_{21}$.) Clearly, $E^*_{21} \in \mathfrak{B}_{21}$. We assert that $E'_{21}\ A^* = E^*_{21}\ A$. Since

$$E'_{nk}\ A^*_{kj} = \left\{ \begin{matrix} 0 \text{ if } (n,k) \notin G \\ E_{nk}\ A_{kj} \text{ if } (n,k) \in G \end{matrix} \right\} = E^*_{nk}\ A_{kj}$$

we have

$$E'_{21}\ A^* = \left\| \sum_{k=1}^{n-1} E_{nk}\ A^*_{k1}, \ldots, \sum_{k=1}^{n-1} E_{nk}\ A^*_{k\ n-1} \right\|$$

$$= \left\| \sum_{k=1}^{n-1} E^*_{nk} A_{k1}, \ \ldots, \ \sum_{k=1}^{n-1} E^*_{nk} A_{kn-1} \right\| = E^*_{21} A.$$

Hence $\left\| \begin{matrix} 0 & 0 \\ E'_{21} & E'_{22} \end{matrix} \right\| \left\| \begin{matrix} A^* & 0 \\ B & C \end{matrix} \right\| = 0$ implies $\left\| \begin{matrix} 0 & 0 \\ E^*_{21} & E'_{22} \end{matrix} \right\| \left\| \begin{matrix} A & 0 \\ B & C \end{matrix} \right\| = 0.$

Define $\mathcal{J} = \{Y_{22} \in \mathfrak{B}_{22} |$ there exists a $Y_{21} \in \mathfrak{B}_{21}$ such that

$$\left\| \begin{matrix} 0 & 0 \\ Y_{21} & Y_{22} \end{matrix} \right\| \left\| \begin{matrix} A & 0 \\ B & C \end{matrix} \right\| = 0\}.$$

Clearly, $\mathcal{J}$ is a left ideal of $\mathfrak{B}_{22}$ containing E'_{22}, hence $\mathfrak{B}_{22} E'_{22} \subset \mathcal{J}$. To obtain the opposite inclusion, let $Y_{22} \in \mathcal{J}$, then there exists a $Y_{21} \in \mathfrak{B}_{21}$ such that

$$\left\| \begin{matrix} 0 & 0 \\ Y_{21} & Y_{22} \end{matrix} \right\| \left\| \begin{matrix} A & 0 \\ B & C \end{matrix} \right\| = 0.$$

Let $Y_{21} = \| Z_{n1}, \ldots, Z_{n\ n-1} \|$. Since $Y_{21} \in \mathfrak{B}_{21}$, $Z_{ni} = 0$ for each i such that $(n,i) \notin G$. Now

$$Y_{21} A^* = \left\| \sum_{k=1}^{n-1} Z_{nk} A^*_{k1}, \ \ldots, \ \sum_{k=1}^{n-1} Z_{nk} A^*_{kn-1} \right\| = Y_{21} A,$$

since $Z_{nk} A^*_{kj} = \begin{cases} 0 & \text{if } (n,k) \notin G \\ Z_{nk} A_{kj} & \text{if } (n,k) \in G. \end{cases}$

Hence, $\left\| \begin{matrix} 0 & 0 \\ Y_{21} & Y_{22} \end{matrix} \right\| \in l_{\mathfrak{A}^*} \left(\left\| \begin{matrix} A^* & 0 \\ B & C \end{matrix} \right\| \right) = \mathfrak{A}^* \left\| \begin{matrix} E'_{11} & 0 \\ E'_{21} & E'_{22} \end{matrix} \right\|$,

and since $\left\| \begin{matrix} E'_{11} & 0 \\ E'_{21} & E'_{22} \end{matrix} \right\|$ is a right identity for $l_{\mathfrak{A}^*} \left(\left\| \begin{matrix} A^* & 0 \\ B & C \end{matrix} \right\| \right)$,

$Y_{22} E'_{22} = Y_{22}$. Therefore, $\mathcal{J} = \mathfrak{B}_{22} E'_{22}$.

Let $Y = E^*_{21}$, then $\begin{Vmatrix} 0 & 0 \\ Y & E_{22} \end{Vmatrix} \begin{Vmatrix} A & 0 \\ B & C \end{Vmatrix} = 0$, i.e.

(*) $YA + E_{22}B = 0$ and $E_{22}C = 0$.

Let $e = \begin{Vmatrix} E & 0 \\ E_{22}Y-YE & E_{22} \end{Vmatrix}$. Now, $\begin{Vmatrix} E & 0 \\ E_{22}Y-YE & E_{22} \end{Vmatrix}^2$

$= \begin{Vmatrix} E^2 & 0 \\ E_{22}YE-YE^2+E_{22}^2Y-E_{22}YE & E_{22}^2 \end{Vmatrix} = \begin{Vmatrix} E & 0 \\ E_{22}Y-YE & E_{22} \end{Vmatrix}$, since $E_{22}^2 = E_{22}$ and $E^2 = E$, i.e., $e^2 = e$. Also, $ex = 0$; from (*) and the definition of E

$$\begin{Vmatrix} E & 0 \\ E_{22}Y-YE & E_{22} \end{Vmatrix} \begin{Vmatrix} A & 0 \\ B & C \end{Vmatrix} = \begin{Vmatrix} EA & 0 \\ E_{22}YA-YEA+E_{22}B & E_{22}C \end{Vmatrix}$$

$= \begin{Vmatrix} 0 & 0 \\ E_{22}(YA+E_{22}B) & 0 \end{Vmatrix} = 0$. Hence, $\mathfrak{A}e \subset l_{\mathfrak{A}}\left(\begin{Vmatrix} A & 0 \\ B & C \end{Vmatrix}\right)$.

To show that this inclusion is in fact an equality, let

$$a = \begin{Vmatrix} Z_{11} & 0 \\ Z_{22} & Z_{22} \end{Vmatrix} \in \mathfrak{A} \text{ such that}$$

$$\begin{Vmatrix} Z_{11} & 0 \\ Z_{21} & Z_{22} \end{Vmatrix} \begin{Vmatrix} A & 0 \\ B & C \end{Vmatrix} = 0.$$

Then, since $l_{\mathfrak{A}}\left(\begin{Vmatrix} A & 0 \\ B & C \end{Vmatrix}\right)$ is a left ideal, we have that

$b = \begin{Vmatrix} 0 & 0 \\ Z_{21} & Z_{22} \end{Vmatrix}$ is in $l_{\mathfrak{A}}\left(\begin{Vmatrix} A & 0 \\ B & C \end{Vmatrix}\right)$, and since $Z_{22} \in \mathfrak{J} = \mathfrak{B}_{22} E_{22}$,

$$c = \left\| \begin{matrix} 0 & 0 \\ Z_{22}Y & Z_{22} \end{matrix} \right\| = \left\| \begin{matrix} 0 & 0 \\ 0 & Z_{22} \end{matrix} \right\| \left\| \begin{matrix} 0 & 0 \\ Y & E_{22} \end{matrix} \right\| \text{ is in}$$

$$l_{\mathfrak{A}} \left(\left\| \begin{matrix} A & 0 \\ B & C \end{matrix} \right\| \right) .$$

Therefore $b-c = \left\| \begin{matrix} 0 & 0 \\ Z_{21}-Z_{22}Y & 0 \end{matrix} \right\| \in l_{\mathfrak{A}} \left(\left\| \begin{matrix} A & 0 \\ B & C \end{matrix} \right\| \right)$.

Thus $\| Z_{21} - Z_{22}\, Y \|\, A = 0$. This implies by our hypothesis, that $\| Z_{21} - Z_{22}\, Y \|\, E = \| Z_{21} - Z_{22}\, Y \|$. Finally, since $Z_{11}\, E = Z_{11}$ and $Z_{22}\, E = Z_{22}$, we have

$$\left\| \begin{matrix} Z_{11} & 0 \\ Z_{21} & Z_{22} \end{matrix} \right\| \left\| \begin{matrix} E & 0 \\ E_{22}Y-YE & E_{22} \end{matrix} \right\| = \left\| \begin{matrix} Z_{11}E & 0 \\ Z_{21}E+Z_{22}E_{22}Y-Z_{22}YE & Z_{22}E_{22} \end{matrix} \right\|$$

$$= \left\| \begin{matrix} Z_{11} & 0 \\ (Z_{21}-Z_{22}Y)E+Z_{22}Y & Z_{22} \end{matrix} \right\| = \left\| \begin{matrix} Z_{11} & 0 \\ Z_{21}-Z_{22}Y+Z_{22}Y & Z_{22} \end{matrix} \right\|$$

$$= \left\| \begin{matrix} Z_{11} & 0 \\ Z_{21} & Z_{22} \end{matrix} \right\| \text{ and the lemma is proved.}$$

4.6 Lemma. Let $\mathfrak{A} = \mathfrak{A}(V_i, J_i, n)$ be a d-ring and assume that each element of $\mathfrak{A}$ can be put in the blocked form

$$\left\| \begin{array}{cc|c} Z_{11} & 0 & 0 \\ 0 & Z_{22} & 0 \\ \hline Z_{31} & Z_{32} & Z_{33} \end{array} \right\| .$$

If the rings

$$\mathfrak{A}_i = \left\| \begin{array}{c|c} Z_{ii} & 0 \\ \hline Z_{3i} & Z_{33} \end{array} \right\| \qquad (i = 1,2)$$

both satisfy condition (8), then so does $\mathfrak{A}$. (The $\mathfrak{B}_{ij}$ referred to in condition (8) are determined in each case by the dotted lines.)

Proof.

$$\text{Let } A = \begin{Vmatrix} A_{11} & 0 \\ 0 & A_{22} \end{Vmatrix} \text{ and let } 1_{\mathfrak{B}_{11}}(A) = B_{11} \begin{Vmatrix} E' & 0 \\ 0 & E'' \end{Vmatrix}.$$

Then $1_{\mathfrak{B}'_{11}}(A_{11}) = \mathfrak{B}'_{11}E'$ where $\mathfrak{B}'_{11}$ is the ring of all blocks Z_{11}, and $1_{\mathfrak{B}''_{11}}(A_{22}) = \mathfrak{B}''_{11}\, E''$, where $\mathfrak{B}''_{11}$ is the ring of all blocks Z_{22}. Let $Y_{21} = \|X_{31}, X_{32}\| \in \mathfrak{B}_{21}$. Now $Y_{21}\, A = 0$ implies $X_{31}\, A_{11} = 0 = X_{32}\, A_{22}$, hence $X_{31}\, E' = X_{31}$ and $X_{32}\, E'' = X_{32}$, by the hypothesis. Therefore,

$$Y_{21}\begin{Vmatrix} E' & 0 \\ 0 & E'' \end{Vmatrix} = \begin{Vmatrix} X_{31} & X_{32} \end{Vmatrix} \begin{Vmatrix} E' & 0 \\ 0 & E'' \end{Vmatrix} = \begin{Vmatrix} X_{31}\, E' & X_{32}\, E'' \end{Vmatrix} = \begin{Vmatrix} X_{31} & X_{32} \end{Vmatrix}$$

and the lemma is proved.

4.7 Definition. A graph G on I_n is said to be connected if and only if for each $i, j \in I_n$ there is a sequence of distinct integers $i = p_1, p_2, \ldots, p_k = j$ such that either (p_{i+1}, p_i) or $(p_i, p_{i+1}) \in G$ $(i = 1, 2, \ldots, k-1)$.

4.8 Lemma. A d-ring $\mathfrak{A} = \mathfrak{A}(V_i, J_i, n)$ is indecomposable if and only if the graph G of $\mathfrak{A}$ is connected.

Proof. Define the relation on I_n by $i \sim j$ if and only if i and j are connected in G. This relation is clearly an equivalence relation. Hence, $I_n = C_1 \cup C_2 \cup \ldots \cup C_t$ where the C_i are mutually disjoint sets called components.

Now suppose G is not connected. Then $t > 1$. Let $G_1 = (C_1 \times C_1) \cap G$ and $G_2 = (C' \times C') \cap G$, where $C' = \cup_{i=2}^{t} C_i$, and define $\mathcal{J}_1 = \sum_{(i,j)\epsilon G_1} e_i\mathfrak{A}e_j$ and $\mathcal{J}_2 = \sum_{(i,j)\epsilon G_2} e_i\mathfrak{A}e_j$.

It can be easily verified, since G is transitive, that $\mathcal{J}_1$ and $\mathcal{J}_2$ are disjoint subrings of $\mathfrak{A}$. Let $(p,q) \epsilon G_2$ and $(i,j) \epsilon G_1$, then $e_p\mathfrak{A}e_q e_i\mathfrak{A}e_j = 0$, if $q \neq i$, and if $e_p\mathfrak{A}e_q e_i\mathfrak{A}e_j \neq 0$, then $e_p\mathfrak{A}e_j \neq 0$, i.e., $(p,j) \epsilon G$. But this implies that either $p \epsilon C_1$ or $j \epsilon C'$. In any case, we have a contradiction. Therefore, multiplication of elements of $\mathcal{J}_2$ by elements of $\mathcal{J}_1$, and conversely, is zero. Therefore $\mathfrak{A} = \mathcal{J}_1 \oplus \mathcal{J}_2$.

To prove the converse, suppose $\mathfrak{A}$ is the direct sum of two non-zero ideals $\mathcal{J}_1$ and $\mathcal{J}_2$. Let $C_p = \{i | e_i\mathfrak{A}e_i \cap \mathcal{J}_p \neq 0\}$ $(p = 1,2)$. Since $e_i\mathfrak{A}e_i$ is isomorphic to $\mathcal{L}(V_i)$, it has a unique minimal non-zero ideal I_o (see [7]). Therefore, since $e_i\mathfrak{A}e_i \cap \mathcal{J}_p$ is an ideal of $e_i\mathfrak{A}e_i$ and $\mathcal{J}_1 \cap \mathcal{J}_2 = 0$, we have that $C_1 \cap C_2 = \emptyset$. This implies, since $e_i \epsilon \mathfrak{A} = \mathcal{J}_1 \oplus \mathcal{J}_2$, that $C_p = \{i | e_i \epsilon \mathcal{J}_p\}$. Now, if $e_i \epsilon J_1$ and $e_j \epsilon \mathcal{J}_2$, then $e_j\mathfrak{A}e_i \cup e_i\mathfrak{A}e_j \subset \mathcal{J}_1 \cap \mathcal{J}_2$. Therefore, $e_i\mathfrak{A}e_j = e_j\mathfrak{A}e_i = 0$, i.e., $(i,j),(j,i) \notin G$. We note that if $i \epsilon C_p$ and (i,j) (or(j,i)) is in G, then $j \epsilon C_p$. Since G is connected, this implies that either $C_1 = I_n$ or $C_2 = I_n$. If $C_p = I_n$, then $1 = e_1 + e_2 + \ldots + e_n \epsilon J_p$. Hence $\mathfrak{A} = \mathcal{J}_p$. We therefore conclude that $\mathfrak{A}$ is irreducible if its graph is connected. This concludes the proof of the lemma.

Let the graph G of a d-ring $\mathfrak{A} = \mathfrak{A}(V_i, J_i, n)$ be connected, let C_1 be a component of I_n and let $C_2 = I_n \setminus C_1$. Let the elements of C_1 and C_2 be ordered according to their numerical value, say

$$C_1 = \{k_1, k_2, \ldots, k_t\} \text{ and } C_2 = \{k_{t+1}, k_{t+2}, \ldots, k_n\}.$$

Define a permutation π on I_n by $i\pi^{-1} = k_i$. Then the elements of $\mathfrak{A}((V_i,\pi),(J_i,\pi),n)$ can be partitioned such that they have the form

$$\begin{Vmatrix} X_{11} & 0 \\ 0 & X_{22} \end{Vmatrix}$$

where X_{11} is a $t \times t$ blocked matrix and X_{22} is a $n-t \times n-t$ blocked matrix (neither X_{11} nor X_{22} is necessarily full).

More generally, we have:

4.9 Corollary (of Proof). **Every d-ring $\mathfrak{A}$ is a direct sum of a finite number of indecomposable d-rings $\mathfrak{A}_i$. Moreover, decompositions for these d-rings may be chosen so that the graphs of $\mathfrak{A}_i$ are just components of the graph of $\mathfrak{A}$.**

Proof. See the proof of 4.8.

4.10 Lemma. Let G be a reflexive, transitive, triangular graph on I_n such that

(i) $(n,i) \in G$ for $i = 1, 2, \ldots, n$,

(ii) G surrounds no zero,

and (iii) $G' = \{(i,j) \in G \mid i,j < n\}$ is connected.

Then $(n-1,i) \in G$ $(i = 1, 2, \ldots, n-1)$.

Proof.

$$\text{If } (j,i),(k,i) \in G' \text{ for } j > k, \text{ then } (j,k) \in G'(J_i, n-1). \tag{9}$$

Now $(n,k),(n,j),(k,i),(j,i) \in G$, hence, by (ii), either (k,j) or (k,j) is in G. Since $j > k$, $(j,k) \in G$, by the assumption that G is triangular. Since $(j,i) \in G'$, $j < n$. Therefore, $(j,k) \in G'$.

We show now that every "column" of G' has at least one non-diagonal element. To accomplish this, we prove:

$$\text{If } 1 \leq i \leq n-1, \text{ then there exists a } k > i \text{ such that } (k,i) \in G'. \tag{10}$$

Since G' is connected for each $i < n-1$, there is a sequence of distinct integers $i = p_1, p_2, \ldots, p_k = n-1$ such that (p_i,p_{i+1}) or (p_{i+1},p_i) is in G' $(i = 1,2,\ldots,k-1)$. We assume that the length of the sequence is minimal. Now, if $(p_2,i) \in G'$, then (10) is established, so we assume $(i,p_2) \in G'$. Either (p_2,p_3) or (p_3,p_2) is in G'. If $(p_2,p_3) \in G'$, then $(i,p_3) \in G'$, since G' is transitive, contradicting the minimality of the sequence. If $(p_3,p_2) \in G'$ and $i > p_3$, then, by (9), $(p_3,i) \in G'$. This proves (10).

By (10), $(n-1,n-2) \in G'$. We proceed by induction as follows: Assume $(n-1,j) \in G'$ for $j = n-1, n-2, \ldots, n-k = i$. By (10), $(k',i-1) \in G'$ for some $k' > i-1$. Now $(n-1,k') \in G'$ since $k' > i$, hence $(n-1,k'),(k',i-1) \in G'$ implies $(n-1,i-1) \in G'$. Therefore $(n-1,j) \in G'$ for $j = n-1, n-2, \ldots, 1$ and the lemma is proved.

4.11 **Lemma.** **Let $\mathfrak{A} = \mathfrak{A}(V_i,J_i,m)$ be a $\mathfrak{d}$-ring with a triangular graph G. If $\mathfrak{B}_{22}$ denotes the ring of all blocks X_{mm} in (6) and $\mathfrak{B}_{11}$ and $\mathfrak{B}_{21}$ are determined accordingly, with**

$\mathfrak{B}_{11}$ Richart, then $\mathfrak{A}$ satisfies condition (8).

Proof. Let $A \in \mathfrak{B}_{11}$ and let $1_{\mathfrak{B}_{11}}(A) = \mathfrak{B}_{11}E$ for some idempotent E in $\mathfrak{B}_{11}$. Let $Y_{21} \in \mathfrak{B}_{21}$ such that $YA = 0$. We must show that $YE = Y$. First, we show that we may assume that none of the blocks on the bottom row of $\mathfrak{A}$ are zero, i.e., $(m,i) \in G$ $(i=1,2,\ldots,m)$. To show this, let k be the largest i such that $(m,i) \notin G$, $(1 \leq k \leq m-1)$. Consider two cases: (i) $k < m-1$ and (ii) $(k = m-1)$. If $k < m-1$, then the elements of $\mathfrak{A}$ have the form

$$\begin{Vmatrix} X_{11} & & & & & \\ \vdots & \ddots & & & & \\ X_{k1} & \cdots & X_{kk} & & & \\ X_{k+1\,1} & \cdots & 0 & X_{k+1\,k+1} & & \\ \vdots & & \vdots & \vdots & \ddots & \\ X_{m1} & \cdots & 0 & X_{m\,k+1} & \cdots & X_{mm} \end{Vmatrix}, \tag{11}$$

for if $(p,k) \in G$ for $p > k$, then $(m,p),(p,k) \in G$ would imply $(m,k) \in G$ which is a contradiction. Now partition the matrices of $\mathfrak{A}$ as follows:

$$\left\Vert \begin{array}{ccc|c} Z_{11} & & & \\ Z_{21} & Z_{22} & & \\ Z_{31} & 0 & Z_{33} & \\ \hline Z_{41} & 0 & Z_{43} & Z_{44} \end{array} \right\Vert \tag{12}$$

where Z_{22} and Z_{44} correspond, respectively, to X_{kk} and X_{mm} in (11). $\mathfrak{B}_{11}$ now consists of the 3×3 block matrices in the upper left hand corner of (12), and $\mathfrak{B}_{22}$ and $\mathfrak{B}_{21}$ are determined accordingly. Let

$$A = \begin{Vmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & 0 & A_{33} \end{Vmatrix}, \qquad E = \begin{Vmatrix} E_{11} & 0 & 0 \\ E_{21} & E_{22} & 0 \\ E_{31} & 0 & E_{31} \end{Vmatrix},$$

$$F = \begin{Vmatrix} I_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{33} \end{Vmatrix} \text{ and } e = \begin{Vmatrix} F & 0 \\ 0 & I_{44} \end{Vmatrix}$$

where I_{ii} is the i th projection on $V = V_1' \oplus V_2' \oplus V_3' \oplus V_4'$

$$(V_1' = \sum_{i=1}^{k-1} \oplus V_i,\ V_2' = V_k,\ V_3' = \sum_{i=k+1}^{m-1} \oplus V_i \text{ and } V_4' = V_m)$$

so that $F \in \mathfrak{B}_{11}$ and $e \in \mathfrak{A}$. Note that $e\mathfrak{A}e$ is the set of all matrices of the form

$$\begin{Vmatrix} \begin{array}{ccc|c} Z_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ Z_{31} & 0 & Z_{33} & 0 \\ \hline Z_{41} & 0 & Z_{43} & Z_{44} \end{array} \end{Vmatrix}$$

which is isomorphic to the d-ring $\mathfrak{A}' = \mathfrak{A}'(\bar{V}_i, \bar{J}_i, 3)$ $(\bar{V}_1 = V_1',\ \bar{V}_2 = V_3',\ \bar{V}_3 = V_4',\ J_1 = \{1\},\ J_2 = \{1,2\}$ and $J_3 = \{1,2,3\})$ of all matrices of the form

$$\begin{Vmatrix} \begin{array}{cc|c} Z_{11} & & \\ Z_{31} & Z_{33} & \\ \hline Z_{41} & Z_{43} & Z_{44} \end{array} \end{Vmatrix}.$$

For the remainder of the proof we identify $e\mathfrak{A}e$ and $\mathfrak{A}'$, hence, it makes sense to speak of the graph G' of $e\mathfrak{A}e$.

We show now that $e\mathfrak{A}e$ has property (8) with $\mathfrak{C}_{ij} = e\mathfrak{A}e \cap \mathfrak{B}_{ij}$ taking the place of $\mathfrak{B}_{ij}$ in the statement of (8). If $m = 1$, then clearly, property (8) holds. Suppose (8) holds for all d-rings whose diagonal blocks number $t < m$. If the above case obtains, i.e., if k is the largest i such that $(m,i) \notin G$, then $e\mathfrak{A}e$ has only $m-1$ diagonal blocks and therefore, has property (8). Let $B = FAF \in \mathfrak{C}_{11}$. Then $1_{\mathfrak{C}_{11}}(B) = \mathfrak{C}_{11}FEF$ and FEF is idempotent, for

$$(FEF)^2 = \begin{Vmatrix} E_{11} & 0 & 0 \\ 0 & 0 & 0 \\ E_{31} & 0 & E_{33} \end{Vmatrix}^2 = \begin{Vmatrix} E_{11} & 0 & 0 \\ 0 & 0 & 0 \\ E_{31} & 0 & E_{33} \end{Vmatrix},$$

since $E^2 = E$, and

$$B = \begin{Vmatrix} A_{11} & 0 & 0 \\ 0 & 0 & 0 \\ A_{31} & 0 & A_{33} \end{Vmatrix} \in \mathfrak{B}_{11}$$

implies that if

$$\begin{Vmatrix} A'_{11} & 0 & 0 \\ 0 & 0 & 0 \\ A'_{31} & 0 & A'_{33} \end{Vmatrix} B = 0$$

then

$$\begin{Vmatrix} A'_{11} & 0 & 0 \\ 0 & 0 & 0 \\ A'_{31} & 0 & A'_{33} \end{Vmatrix} = \begin{Vmatrix} A''_{11} & 0 & 0 \\ A''_{21} & A''_{22} & 0 \\ A''_{31} & 0 & A''_{33} \end{Vmatrix} \begin{Vmatrix} E_{11} & 0 & 0 \\ E_{21} & E_{22} & 0 \\ E_{31} & 0 & E_{33} \end{Vmatrix},$$

hence

$$\begin{Vmatrix} A'_{11} & 0 & 0 \\ 0 & 0 & 0 \\ A'_{31} & 0 & A'_{33} \end{Vmatrix} = \begin{Vmatrix} A''_{11} & 0 & 0 \\ 0 & 0 & 0 \\ A''_{31} & 0 & A''_{33} \end{Vmatrix} \begin{Vmatrix} E_{11} & 0 & 0 \\ 0 & 0 & 0 \\ E_{31} & 0 & E_{33} \end{Vmatrix} .$$

Now observe that if $(m,i) \in G$, then $(m,i) \in G'$, i.e., the bottom row of blocks of $\mathfrak{A}$ is the same as that of $e\mathfrak{A}e$. Thus $Y \in \mathfrak{B}_{21}$ implies $Y \in \mathfrak{C}_{21}$, and by (12),

$$\| Y = \ Z_{41} \ 0 \ Z_{43} \| ,$$

hence $YA = 0$ implies $YB = 0$. Therefore, $YE = YFEF = Y$.

If $(m,m-1) \notin G$, then $\mathfrak{A}$ has the form

$$\begin{Vmatrix} Z_{11} & 0 & 0 \\ Z_{21} & Z_{22} & 0 \\ Z_{31} & 0 & Z_{33} \end{Vmatrix}$$

where Z_{22} and Z_{33} correspond to $X_{m-1\ m-1}$ and X_{mm}, respectively. If we set

$$A = \begin{Vmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{Vmatrix} , \qquad E = \begin{Vmatrix} E_{11} & 0 \\ E_{21} & E_{22} \end{Vmatrix}$$

$$F = \begin{Vmatrix} I_{11} & 0 \\ 0 & 0 \end{Vmatrix} \quad \text{and} \quad e = \begin{Vmatrix} F & 0 \\ 0 & I_{44} \end{Vmatrix} ,$$

then the same argument above yields $YE = Y$.

For the remainder of the proof, assume $(m,i) \in G$, $(i = 1, 2, \ldots, m)$. Now, by Lemmas 4.6 and 4.8 we may assume that the d-ring $\mathfrak{B}_{11}$ is indecomposable.

Since G is triangular, the hypotheses of Lemma 4.10 are satisfied. Therefore, $(m-1,i) \in G$ $(i = 1, 2, \ldots, m-1)$. Now, there are no zero blocks on the last two rows of $\mathfrak{A}$ and the elements of $\mathfrak{A}$ have the form

$$\left\|\begin{array}{cc|c} Z_{11} & 0 & 0 \\ Z_{21} & Z_{22} & 0 \\ \hline Z_{31} & Z_{32} & Z_{33} \end{array}\right\|$$

where the blocks Z_{21} and Z_{31} are full matrix blocks over D. $\mathfrak{B}_{11}$ is the set of all 2 x 2 matrix blocks in the upper left hand corner. Now let $YA = 0$ for some $Y = \left\|Z_{31} \quad Z_{32}\right\|$ and let y be, literally, a row of Y. Since $\mathfrak{B}_{21}$, the set of all blocked matrices $\left\|Z_{31} \quad Z_{32}\right\|$ over D, is full, let

$$Y_o = \left\|\begin{array}{cc} X_{11} & 0 \\ X_{21} & X_{22} \end{array}\right\| \in \mathfrak{B}_{11}$$

be such that $X_{11} = 0$, one row of $\left\|X_{21} \quad X_{22}\right\|$ is the same as y and every other row is zero. Clearly, $Y_oA = 0$. Hence $Y_oE = Y_o$. Since this holds for any row of Y, $YE = Y$ and the proof is complete.

We are now armed with enough lemmas to prove our main result.

4.12 Theorem. Let $\mathfrak{A} = \mathfrak{A}(V_i, J_i, m)$ be a d-ring with graph G. Then $\mathfrak{A}$ is Rickart if and only if G surrounds no zero.

Proof. We assume by our previous comments that G is triangular. To prove the necessity, let $\mathfrak{A}$ be Rickart and suppose G surrounds a zero. Then, by Lemmas 4.4 and 4.4' there is an idempotent e in $\mathfrak{A}$ such that $e\mathfrak{A}e$ is not Rickart.

But, by Lemma 4.3, $e\mathfrak{A}e$ is Rickart which is a contradiction. Therefore, G surrounds no zero.

For the converse we proceed by induction on the number m of diagonal blocks, i.e, the number of elements in the vertex set I_m of G.

If $m = 1$, then $\mathfrak{A} = \mathfrak{L}(V)$ which is Baer (see [13]), hence Rickart. Suppose that the theorem is true for all $k < m$ and let $\mathfrak{A}$ be as in the hypothesis. Since G is triangular, $\mathfrak{A}' = \mathfrak{A}'(V_i, J_i, m-1)$ is a d-ring (see Chapter III, p.28) with graph $G' = G'(J_i, m-1)$ which is also triangular. Since G surrounds no zero, G' surrounds no zero, and by the inductive hypothesis $\mathfrak{B}_{11}$ is Rickart. Let $\mathfrak{B}_{22}$ be the full ring of all blocks X_{mm} and let $\mathfrak{B}_{21}$ be determined accordingly. By Lemma 4.11, $\mathfrak{A}$ has property (8). Finally, by Lemma 4.5, $\mathfrak{A}$ is Rickart and the theorem is established.

4.13 Corollary (Clark). Let $\mathfrak{A} = \mathfrak{A}(V_i, J_i, n)$ be a d-ring with graph G. If V is finite dimensional, then $\mathfrak{A}$ is Baer if and only if G surrounds no zero.

Proof. The necessity follows directly from Theorem 4.12.

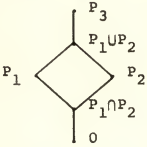
For the converse, we need to prove that the left annihilator of any subset S of A has the form $\mathfrak{A}e$, for some $e^2 = e \in \mathfrak{A}$. By the results of Maeda [6], it suffices to assume, since V is finite dimensional, that $S = \{\sigma\}$. The corollary now follows from Theorem 4.12.

Recall from Chapter II that a d'-ring $\mathfrak{A} = \mathfrak{A}(J)$ is a d-ring $\mathfrak{A} = \mathfrak{A}(V_i, J_i, n)$ if J is finite and distributive.

Under the assumption that J is finite and distributive, Johnson [2] proves that $\mathfrak{A}$ is non-Baer if J satisfies the following additional condition:

> J has three irreducible elements P_1, P_2 and P_3 such that $P_1 \cup P_2 \subset P_3$ and $P_1 \cap P_2$ is different from 0, P_1 and P_2. (13)

Evidently condition (13) is satisfied if and only if the lattice $L(V_{\mathfrak{A}})$ of invariant subspaces of V has a sublattice isomorphic to the following lattice:



where P_1, P_2, P_3 are irreducible elements.

We now show that (13) is equivalent to:

The graph G of $\mathfrak{A}$ surrounds a zero. (14)

We have shown that the lattice $L(\mathfrak{A})$ of subsets of I_n associated with a d-ring $\mathfrak{A}$ is isomorphic to J (see p. 18). Let $[i]$, $[j]$, $[k]$ and $[h]$ be the irreducible elements of $L(\mathfrak{A})$ corresponding to P_1, P_2, P_3 and $P_1 \cap P_2$ of condition (13) under the isomorphism $L(\mathfrak{A}) \cong J$. Now, $0 \neq [h] \subset [i] \cap [j]$, $[i] \cup [j] \subset [k]$ and, since $[k]$ is irreducible, $[i] \cup [j] \neq [k]$. Hence

$\{(h,h),(i,i),(j,j),(k,k),(i,h),(j,h),(k,h),(k,i),(k,j)\} \subset G$.

However, $(i,j) \notin G$, otherwise $[j] \subset [i]$, by Lemma 2.5, i.e. G surrounds a zero. The argument is reversible, hence,

(13) is equivalent to (14). Johnson's theorem can now be stated:

4.14 *Theorem.* If the graph G of a d-ring $\mathfrak{A} = \mathfrak{A}(V_i, J_i, n)$ surrounds a zero, then $\mathfrak{A}$ is non-Baer.

Theorem 4.12 can be restated as follows:

4.12' *Theorem.* A d-ring $\mathfrak{A}$ of linear transformations on a vector space V is Rickart if and only if its lattice J of invariant subspaces does not satisfy (13).

We conclude this manuscript by noting three open questions:

(1) If $\mathfrak{A}$ is a d-ring, then perhaps the most obvious question to ask is: In order for $\mathfrak{A}$ to be a Baer ring, is it necessary, *and sufficient*, that the graph G of $\mathfrak{A}$ surround no zero?

(2) Let $\mathcal{R}_I(\mathfrak{A})$ ($\mathcal{L}_I(\mathfrak{A})$) be the set of all principal right (left) ideals generated by idempotents, partially ordered by set inclusion. In [6], Maeda shows that if $\mathfrak{A}$ is Rickart, then $\mathcal{R}_I(\mathfrak{A})$ ($\mathcal{L}_I(\mathfrak{A})$) is a relatively semi-orthocomplemented lattice. Furthermore, if the lattice $\mathcal{R}_I(\mathfrak{A})$ is complete, then $\mathfrak{A}$ is a Baer ring. Now the question is: Can one give necessary and sufficient conditions on a lattice L in order that L be isomorphic to $\mathcal{R}_I(\mathfrak{A})$ for some d-ring $\mathfrak{A}$ of linear transformations.

(3) Is it possible to give a constructive description of those finite, transitive, reflexive, directed graphs which surround no zeros?

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BIOGRAPHICAL SKETCH

Charles Karter Martin was born on August 4, 1935, in Fayetteville, Tennessee. He was raised in Chicago, Illinois, but divided the last two secondary school years between Birmingham, Alabama,and Clearwater, Florida.

Having passed the summer of 1953 as a professional baseball player in Abbyville, Louisiana, he joined the United States Coast Guard where he served on a weather ship during the years 1953 through 1957. In October of 1956, he was married to Miss Shirley Ann Fricker of New Bedford, Massachusetts. He entered the University of Florida in September, 1958, and was graduated with honors in June of 1962. Having received a Bachelor of Arts degree in Mathematics, he remained at the University of Florida where he has pursued his work toward the degree of Doctor of Philosophy. He held a one-half time teaching assistantship from 1962 through 1964 and was an interim instructor during the 1965-66 academic year. He is a member of the American Mathematical Society and the Mathematical Association of America.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

August 13, 1966

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